Creating a Polynomial Function to Fit a Table

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Creating a Polynomial Function to Fit a Table* Illustration: This Illustration's student dialogue shows the conversation among three students who are looking for a rule that defines a table of values. While at first students jump to a linear rule that works, they also explore non-linear possibilities and come up with a method for generating infinitely many functions that fit the table values.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 9–10

Target Content Domain
Interpreting Functions (Functions Conceptual Category)
Building Functions (Functions Conceptual Category)
Arithmetic with Polynomials and Rational Expressions (Algebra Conceptual Category)

Highlighted Standard(s) for Mathematical Content

F.IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

F.BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

A.APR.B.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Math Topic Keywords: functions, tables, roots, zeros

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Find a rule that agrees with this table:

x	$f(x)$
1	5
2	8
3	11
4	14

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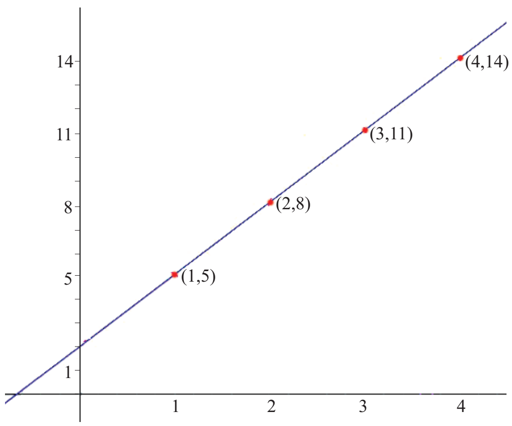
Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

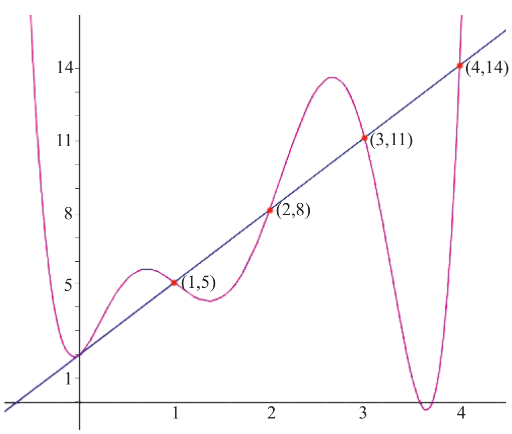
Students in this dialogue have been studying functions and are learning how to write functions based on different types of information.

(1) Chris: Well, duh, that's obvious. f goes up by 3, so…

(2) Lee: So $f(x) = 3x + 2$. *[sketches the following]*



(3) Matei: But couldn't we also get a graph that looks like this? *[draws the following]*



See? When $x = 1$, my function would output 5, and so on!

(4) Chris: Well, sure, you can draw a squiggle, but can you define a *function* that has a graph that looks like that? Well, I guess that *is* a function. But can you write an algebraic definition for a mess like that?

(5) Matei: We'd need some different function g so that $g(1)=5$, and $g(2)=8$, and so on.

(6) Lee: And we know that $f(1)=5$, and $f(2)=8$, so…

(7) Chris: So, doesn't that mean that they're *not* different?

(8) Matei: Well, they're not different *at those points*. In other words, the *difference* between f and g is 0 when x is 1, 2, 3, and 4.

(9) Chris: So, we're looking for a way to change $f(x)$ into $g(x)$ while keeping $f(x)$ and $g(x)$ the same at those four values of x?

(10) Lee: Yeah, so that means we need to add something to $f(x)$, but whatever we add must be 0 at those points.

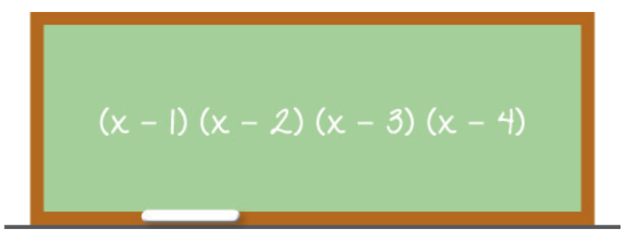
(11) Chris: Adding 0 to $f(x)$ just gives us $f(x)$, and that's no help at all. If it's 0 at *those* points, isn't it just 0 everywhere?

(12) Matei: Well, no, not necessarily. We know how to write functions that are 0 in some places and not others. The question is just whether we can write one that is 0 at *those* exact places, at $x=$ 1, 2, 3, and 4.

(13) Lee: Ah. If we write a function that's *zero* at $x=$ 1, 2, 3, and 4 and add that to our f, the resulting function will have the same values as f at $x=$ 1, 2, 3, and 4. That's the g we're looking for! So, how can we write a function that's 0 at those places?

[Students work individually for 15 minutes and test various functions, looking to find one that is 0 for inputs of 1, 2, 3, and 4.]

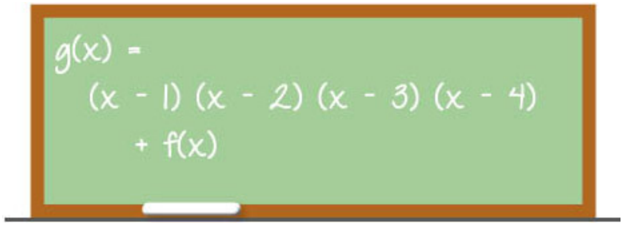
(14) Matei: Got it! Here's an expression that should evaluate to 0 when x is 1, 2, 3, or 4. *[writes]*



(15) Lee: How'd you get that?

(16) Matei: I know that if $x=1$ then $(x-1)=0$, and so $(x-1)$ *times* anything will also…

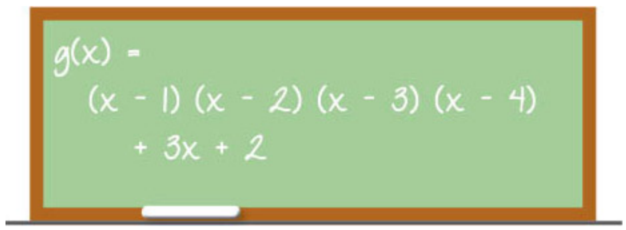
(17) Chris: Oh! And so… *[writes]*

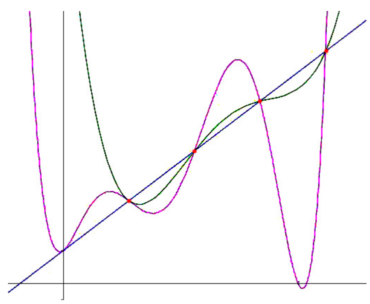


Woohoo! We did it!!!

(18) Lee: So, we just plug in $f(x)$. Yay!

(19) Chris: So $g(x)$ equals… *[writes the function on the board and graphs* $g(x)$ *in green]*

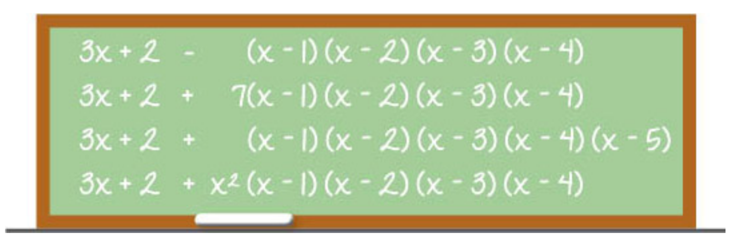


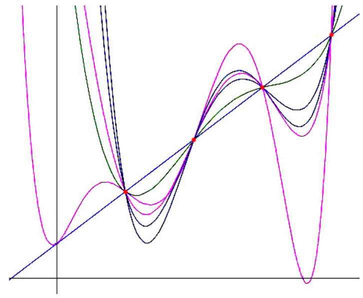


But *that's* not the graph you sketched, Matei! It's a different one!

(20) Matei: Who cares?! It works.

(21) Lee: In fact, there's more! Since $(x-1)(x-2)(x-3)(x-4)$ is 0 in all the places we care about, any multiple of it will work just as well. So we could also use any of these: *[writes some expressions and graphs several of them]*





(22) Chris: OK, we only needed one rule, and now we've got six! But all our new ones are based on Matei's original idea of $(x-1)(x-2)(x-3)(x-4)$. I wonder if there are any *other* ways, *different* ways, to build a new rule that fits that table!

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. By line 2, the students in the dialogue have solved the original problem: "Find a rule that agrees with this table," but Matei (line 3) extends the problem. Matei is problem-posing by varying some aspect of the problem. Matei is also, in effect, asking about the uniqueness of their solution. "Going beyond" a problem in this sense is not directly addressed in any of the MPs. Which MPs seem to indirectly address this idea?

3. None of the students say, "Come on, Matei, we've solved this problem. Let's just move on to the next one!" Instead, they take on the new problem that Matei poses. How (if at all) is that disposition captured in the mathematical practice standards?

4. While students are building functions, why isn't this dialogue an example of MP 4: Model with mathematics?

5. What is the lowest degree of a non-linear polynomial that will fit all the entries in the original table?

6. Lee's statement in line 10 implies that the only way to solve this problem is to add to $f(x)$ another function that equals 0 at the indicated values. Could a different approach generate functions that agree with the table in the task?

7. In your own classroom, how do you discuss the fact that, for any data set, there is always more than one model, and that some judgment goes into the decision about which model is "right" for your purpose? How can these conversations be started and/or facilitated in math classes?

8. After struggling for a while, Matei comes up with an expression that is 0 for the inputs given in the original task's table (line 14). If nobody had come up with the expression $(x-1)(x-2)(x-3)(x-4)$, how might you help someone arrive at this idea?

9. In line 16, Matei is interrupted mid-sentence. Following Matei's line of reasoning, how might Matei reasonably complete that statement about $(x-1)$ times anything?

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10. In line 17, Chris makes a leap. How can Chris be sure that $g(x) = (x-1)(x-2)(x-3)(x-4) + f(x)$ is the function they are looking for: a new function, different from $f(x)$, that still fits the table?

11. In line 11, Chris proposes adding 0. In line 17, Chris instead adds $(x-1)(x-2)(x-3)(x-4)$. Graph *those* two functions—$h_1(x) = 0$ and $h_2(x) = (x-1)(x-2)(x-3)(x-4)$—and describe where they agree?

12. The students conclude that $f(x) + \underbrace{\text{anything}} \cdot (x-1)(x-2)(x-3)(x-4)$ would agree with the table when $x =$ 1, 2, 3, and 4. Anything?! Really? How might you prove or disprove such a claim? What can you say about agreement (or lack of agreement) at other values of x?

13. The students conclude that starting with f and adding some h that has 0s at $x =$ 1, 2, 3, and 4 will have the effect that they want, because it does not mess up the values of f at any of the critical places. Instead of *adding 0 to* f, they might equally well have thought of *multiplying* f *by 1*, finding some function h such that $h(x) = 1$ when $x =$ 1, 2, 3, and 4. Come up with some examples of that kind.

14. Show that the "multiply by 1" strategy is really redundant. That is, show that multiplying by 1 does not produce any solutions that can't be produced with the "add 0" strategy.

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Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

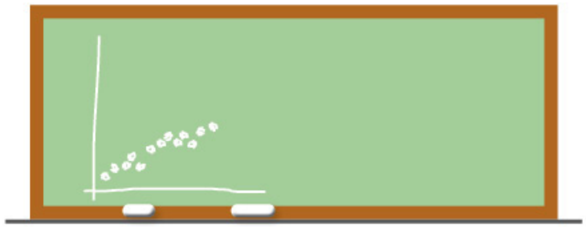
Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	In this dialogue, students are making sense of a problem that Matei poses (line 3), namely the possibility that there is more than one function that fits the table they were given. By "consider[ing] analogous problems"—the various possible graphs and functions—students "gain insight into [the problem's] solution" and learn that there is a wide range of solutions and not just the immediately obvious one of $f(x) = 3x + 2$. Also, students clearly understand the constraints of the problem—that their function needs to fit only the values given in the table, and all other values are constraint-free. They introduce the word "points" early and show comfort in switching between equations, tables, and graphs and establishing the correspondences.
 Use appropriate tools strategically.	Students make graphs to hypothesize alternative solutions (lines 2–3). They also use graphs to check if newly generated functions meet the requirements of the problem (line 19) and to explain why there are infinitely many solutions (line 21).
 Look for and make use of structure.	Students look for and make use of structure in several ways. In developing $(x-1)(x-2)(x-3)(x-4)$ as a polynomial that will modify f in the way they want, students show an understanding of the role of the 0 factors in their polynomial. They further understand that even more complicated expressions will serve to produce a function that agrees with the table in the task as long as they contain that structure as one factor.

Commentary on the Mathematics

When school problems (or tests) present a sequence of numbers like 5, 8, 11, 14, and ask "What is the pattern?" or "What number comes next?" they are presenting a very similar task to the one posed to the students in this dialogue. "Find a rule that agrees with a table whose first entry is 5, whose second entry is 8, whose third entry is 11…" As Chris, Lee, and Matei discover, though, the answer depends on how deep you're willing to look. The rule that Chris calls "obvious" in line 1 is "add 3 to get the next number" and so Chris and Lee choose $f(x) = 3x + 2$ as The Rule. If asked what number would come next, they would no doubt predict 17. Matei's idea might cost points on school tests but shows real mathematical insight! While the *simplest* rule is “add 3,” that is not the *only* rule that generates those numbers.

Any time we are modeling data—whether finding an *exact* fit as is suggested (and possible) in this problem or finding a *best* fit as is more common with measurement data—it is important to keep in mind that many models are possible, and some judgment is required about what model makes the most sense. For example, we might reasonably look for a line of best fit for the following data set…



…but a quadratic would certainly fit it better, and an even better model might be based on $x^{\frac{1}{2}}$ or a log function. In fact, a polynomial of degree 12 could fit it perfectly, wiggling its way along and passing precisely through every single data point! That doesn’t mean that a polynomial of degree 12 is a better model of the *phenomenon* from which the data set was drawn. In fact, such a fit would almost certainly be a terrible model. Infinitely many models are *always* possible; the decision of which model is “right” requires understanding the context, not just how to fit functions to points.

The students’ initial problem, though, gives no context for the data—it isn’t modeling in the usual sense—and so it’s reasonable for students to assume that “find a rule that agrees with this table” calls for *exact* agreement, and it’s even reasonable to assume it means “simplest rule.” Matei is really posing a new problem: Matei is asserting that there is no *unique* rule. Without being explicit and precise about it, Matei is inviting the group to explore what variation in rules might be possible, or the commonalities that *all* rules have.

> The functions $f(x) = 3x + 2$, $g(x) = 3x + 2 + (x-1)(x-2)(x-3)(x-4)$ and $d(x) = 3x + 2 - (x-1)(x-2)(x-3)(x-4)$ generate these tables:

x	$f(x)$
1	5
2	8
3	11
4	14
5	17

x	$g(x)$
1	5
2	8
3	11
4	14
5	41

x	$d(x)$
1	5
2	8
3	11
4	14
5	–7

(That’s equivalent to saying that we can just as logically justify following the sequence 5, 8, 11, 14 by 41 or –7 as by 17.)

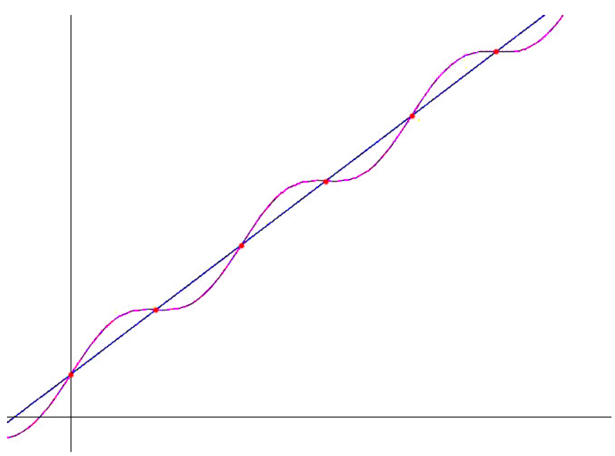
Starting with the simple rule, one can *add* any function that does not mess up the values at any of the given values, so such a function must have the value 0 at all those places. In their work, students crafted (but never named) a function $h(x) = (x-1)(x-2)(x-3)(x-4)$ as an example of such a function. Also, because its value is 0 at the critical places, any multiple of it will also have the value 0 at those places. They concluded that $f(x)+\underbrace{\text{anything}}\cdot h(x)$ would agree with the table, because anything—any constant, any function (as long as it is defined at the necessary places)—times 0 will still be 0, and h guarantees the necessary 0s.

The alternative solutions that the students play with are all polynomial functions. They all agree with their original $f(x)$ at $x =$ 1, 2, 3, and 4 and a maximum of n places where n is the degree of the polynomial solution they've found. If the new g is restricted to being a *polynomial* function of the form $f+h$, then $h(x)$ is a polynomial function with 0s at 1, 2, 3, 4.

The students only thought of polynomial solutions that involve *adding 0*. Their reasoning would have been equally good if they had suggested *multiplying by 1*, e.g., $f(x)\cdot(1+h(x))$, where h is any function that has 0s at $x =$ 1, 2, 3, and 4. This solution is equivalent. Multiplying $f(x)\cdot(1+h(x))$ out, we get $f(x)+f(x)\cdot h(x)$ (with the same restrictions on h) which is included in $\underbrace{\text{anything}}\cdot h(x)$.

In fact, we can show that this is a complete characterization of polynomials that agree with $f(x)=3x+2$ at $x =$ 1, 2, 3, and 4. For f to agree with g at the four required points, $f-g$ must be 0 at those points. Call $f-g$ a new function h; then h has 0s at those four values x. By the factor theorem, therefore, h is divisible by $(x-1)$, $(x-2)$, $(x-3)$, and $(x-4)$. Because these are relatively prime, it is also divisible by their product. So all *polynomial* functions that agree with $f(x)=3x+2$ at 1, 2, 3, and 4 must have the form $3x+2+k(x)[(x-1)(x-2)(x-3)(x-4)]$.

Two *different* polynomial functions can agree in a finite tabulation and disagree everywhere else. Non-polynomial functions can agree with f in *infinitely* many places, and still be distinct from f. For example, note that $\sin(\pi x)=0$ for all integer values of x (in radians), so $g(x)=3x+2+\sin(\pi x)$ will agree with $f(x)=3x+2$ for all integer values of x, as this graph illustrates.



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<u>Evidence of the Content Standards</u>

In the dialogue, students are comparing the inputs-outputs of functions described in various ways, as described in F.IF.C.9. Students are comparing the input-outputs of functions written as tables, graphs, and equations. In coming up with the algebraic equation for a non-linear function that fits the tabular data, students understand that their new function will take the form of "$f(x)+\text{something}$" where the "something" must be 0 for particular inputs; this extends the idea in F.BF.B.3, which focuses only on adding constants: students in this dialogue add an algebraic expression to their linear function. Also, once students identify the "something" as $(x-1)(x-2)(x-3)(x-4)$, they realize that including an extra factor to that expression will still produce a function that has 0s in the right places, and so, when added to f, solves the problem. Lastly, developing the expression $(x-1)(x-2)(x-3)(x-4)$ is based on the students' understanding that the roots of a polynomial may be written as factors (A.APR.B.3).

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

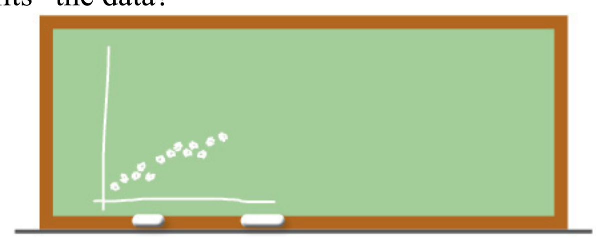
1. What kind of random squiggle can be the graph of a function from x to y? What kinds of squiggles cannot be graphs of functions?

2. In line 16 of the student dialogue, Matei is interrupted mid-sentence. Write the rest of what Matei seemed to be trying to say.

3. Make up 3 more functions that fit the data set and check.

4. The students conclude that starting with f and adding some h that has zeros at $x =$ 1, 2, 3, and 4 will have the effect that they want, because it does not mess up the values of f at any of the critical places. Instead of *adding 0 to* f, they might equally well have thought of *multiplying* f *by 1*, finding some function h such that $h(x) = 1$ when $x =$ 1, 2, 3, and 4. Come up with some examples of that kind.

5. Given a scatter plot like the one below, give two possible interpretations of what it means to find a function that "fits" the data?



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Related Mathematics Tasks

1. Find three different functions (of any kind) that fit through $(1,9)$, $(2,5)$, $(3,1)$, $(4,-3)$.

2. Find three different functions (of any kind) that fit through $(1,1)$, $(2,3)$, $(3,6)$, $(4,10)$.

3. Fill in the input-output table below for the function, $f(x)=\frac{5}{2}(x-1)(x-2)$.

x	$f(x)$
0	
1	
2	
3	

4. A. Given the input-output table below, write a function that includes all the points.

x	$g(x)$
1	–7
2	0
3	0

B. Write a function for each of the input-output tables below.

x	$h(x)$
1	0
2	–1
3	0

x	$j(x)$
1	0
2	0
3	3

C. Fill in the input-output table for $q(x)=g(x)+h(x)+j(x)$ given the input-output tables in parts A and B. Also write a function for $q(x)$ given the previous parts.

5. By first separating the table below into three input-output tables, write a function for $f(x)$.

x	$f(x)$
1	5
2	–2
3	8

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. By line 2, the students in the dialogue have solved the original problem: "Find a rule that agrees with this table," but Matei (line 3) extends the problem. Matei is problem-posing by varying some aspect of the problem. Matei is also, in effect, asking about the uniqueness of their solution. "Going beyond" a problem in this sense is not directly addressed in any of the MPs. Which MPs seem to indirectly address this idea?

   "Making sense" of a mathematical problem often involves exploring the surround, generalizing, or extending the problem. The MP 1 statement that "[mathematically proficient students] consider analogous problems" is set in a context that seems to justify doing so primarily "to gain insight into [the] solution" of "the original problem," but the problem that Matei poses leads to insights about a broader problem, of fitting curves to data. By questioning the uniqueness of solutions, Matei is also exploring the precision (MP 6) of the original problem statement. "Find a rule" doesn't precisely specify the generality or narrowness of the rule: Matei rejects the "surface" interpretation of the question to pursue a more general solution.

3. None of the students say, "Come on, Matei, we've solved this problem. Let's just move on to the next one!" Instead, they take on the new problem that Matei poses. How (if at all) is that disposition captured in the mathematical practice standards?

   Again, this kind of depth is not explicitly mentioned in the standards but is best captured in the "make sense" and "persevere" ideas in MP 1.

4. While students are building functions, why isn't this dialogue an example of MP 4: Model with mathematics?

   On the surface, the problem might appear to involve modeling, in that students are fitting curves to data, but MP 4 is about using mathematics to capture some aspect of a phenomenon; this problem presents no phenomenon. In fact, that is one reason why many curves "fit" the data. If the given table did represent some real-world situation, then assumptions about the phenomenon would help determine whether the "right" model should be linear or something else.

5. What is the lowest degree of a non-linear polynomial that will fit all the entries in the original table?

   Provided no two points are directly above one another (no two on the same vertical line), a polynomial of degree 1 (linear) or higher is sufficient to pass through any two points. A polynomial of degree 2 (quadratic) or higher is sufficient to pass through any three points (though if the three points are collinear, the point set can be fit perfectly by a polynomial of lower degree). In general, a polynomial of degree n or higher is sufficient to pass through any $(n+1)$ points (though, again, special point sets may be fit perfectly by polynomials of lower degree). For this data set, the lowest degree polynomial, other than linear, that will fit is a degree 3 (cubic) polynomial.

6. Lee's statement in line 10 implies that the only way to solve this problem is to add to $f(x)$ another function that equals 0 at the indicated values. Could a different approach generate functions that agree with the table in the task?

   While the approach Lee suggests does work, it is not the only one available. An alternative approach can be seen in question 13.

7. In your own classroom, how do you discuss the fact that, for any data set, there is always more than one model, and that some judgment goes into the decision about which model is "right" for your purpose? How can these conversations be started and/or facilitated in math classes?

   Discuss with colleagues to share ideas.

8. After struggling for a while, Matei comes up with an expression that is 0 for the inputs given in the original task's table (line 14). If nobody had come up with the expression $(x-1)(x-2)(x-3)(x-4)$, how might you help someone arrive at this idea?

   You might ask "How can you write a function with roots at $x=$ 1, 2, 3 and 4?" Hearing the word "roots" might be enough to suggest the idea of writing factors. If four roots is too challenging, you might try an analogous case with only two roots. Since Algebra I students have a lot of experience factoring quadratics in order to find roots, asking students to find a function with roots at only 1 and 2 (for example) may be easier. Alternatively, you might ask students to find the roots of x^2+5x+6 and then work backwards to see how they could generate the polynomial from the roots.

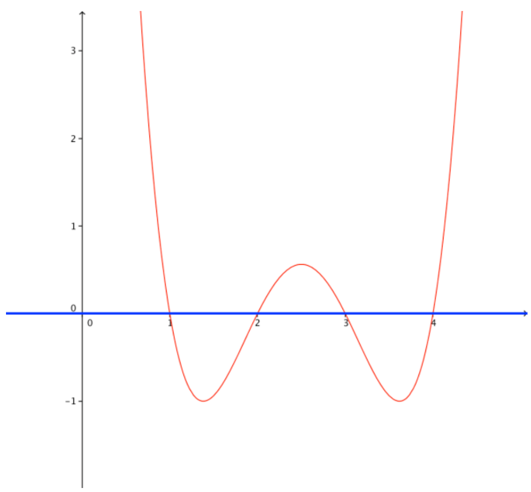
9. In line 16, Matei is interrupted mid-sentence. Following Matei's line of reasoning, how might Matei reasonably complete that statement about $(x-1)$ times anything?

   …and so $(x-1)$ times anything will also be 0.

10. In line 17, Chris makes a leap. How can Chris be sure that $g(x)=(x-1)(x-2)(x-3)(x-4)+f(x)$ is the function they are looking for: a new function, different from $f(x)$, that still fits the table?

    When $x=1$, $(x-1)$ is 0, so $(x-1)$ times anything—in particular the other factors in the expression $(x-1)(x-2)(x-3)(x-4)$ that they've generated—must be 0. We can use the same reasoning when $x=$ 2, 3, or 4: the expression evaluates to 0 in those cases, as well. So, at least at those four values of x, we know that $f(x)+(x-1)(x-2)(x-3)(x-4)=f(x)+0$, and, therefore, it also fits the table.

11. In line 11, Chris proposes adding 0. In line 17, Chris instead adds $(x-1)(x-2)(x-3)(x-4)$. Graph *those* two functions—$h_1(x)=0$ and $h_2(x)=(x-1)(x-2)(x-3)(x-4)$—and describe where they agree?



    $h_1(x)$ is graphed in blue and $h_2(x)$ is graphed in red. They agree when x is 1, 2, 3, and 4.

12. The students conclude that $f(x)+\underbrace{\text{anything}}\cdot(x-1)(x-2)(x-3)(x-4)$ would agree with the table when $x=$ 1, 2, 3, and 4. Anything?! Really? How might you prove or disprove such a claim? What can you say about agreement (or lack of agreement) at other values of x?

    Yes, anything, as long as it remains defined for $x=$ 1, 2, 3, and 4. Whether we multiply $(x-1)(x-2)(x-3)(x-4)$ times –7 or $\frac{\pi}{31}$ or $\cos(x^2)$ or $(x-5)$, the result will be 0 when $x=$ 1, 2, 3, and 4 because $(x-1)(x-2)(x-3)(x-4)$ itself is 0 at those values of x. At other values of x, nothing can be said without knowing more.

13. The students conclude that starting with f and <u>adding</u> some h that has 0s at $x=$ 1, 2, 3, and 4 will have the effect that they want, because it does not mess up the values of f at any of the critical places. Instead of *adding 0 to* f, they might equally well have thought of

multiplying f *by 1*, finding some function h such that $h(x)=1$ when $x=$ 1, 2, 3, and 4. Come up with some examples of that kind.

Perhaps the simplest (non-trivial) one is $h(x)=(x-1)(x-2)(x-3)(x-4)+1$. There are infinitely many others, including $h(x)=x^2(x-1)(x-2)(x-3)(x-4)+1$ or $h(x)=(e^x+5)(x-1)(x-2)(x-3)(x-4)+1$.

14. Show that the "multiply by 1" strategy is really redundant. That is, show that multiplying by 1 does not produce any solutions that can't be produced with the "add 0" strategy.

For a single case,

$$f(x)[\underbrace{\text{anything}}\cdot(x-1)(x-2)(x-3)(x-4)+1]=$$

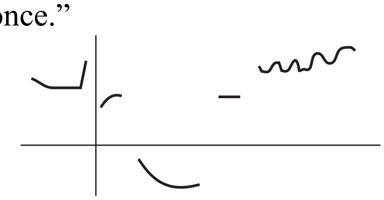
$$f(x)\cdot\underbrace{\text{anything}}\cdot(x-1)(x-2)(x-3)(x-4)+f(x)$$

which is of the form $f(x)+\underbrace{\text{anything}}\cdot(x-1)(x-2)(x-3)(x-4)$. A little work can generalize this to cases where the multiplier does not initially look like $[\underbrace{\text{anything}}\cdot(x-1)(x-2)(x-3)(x-4)+1]$.

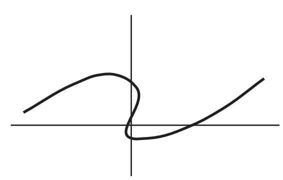
Possible Responses to Student Discussion Questions

1. What kind of random squiggle can be the graph of a function from x to y? What kinds of squiggles cannot be graphs of functions?

Any squiggle, dotted line, patchwork of paths represents a function from x to y as long as it represents a mapping in which every chosen x value has no more than one y value associated with it. This is sometimes called the "vertical line test" because an equivalent statement is "at every point on the x axis, a line drawn vertically intersects the graph no more than once."



Strange, but a function



Tame, but not a function

2. In line 16 of the student dialogue, Matei is interrupted mid-sentence. Write the rest of what Matei seemed to be trying to say.

…and so $(x-1)$ times anything will also be 0.

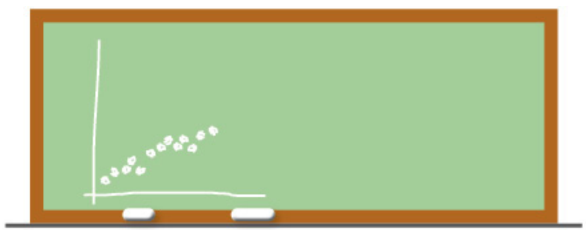
3. Make up 3 more functions that fit the data set and check.

   Anything of the form $3x + 2 + \text{anything} \cdot (x-1)(x-2)(x-3)(x-4)$.

4. The students conclude that starting with f and <u>adding</u> some h that has zeros at $x =$ 1, 2, 3, and 4 will have the effect that they want, because it does not mess up the values of f at any of the critical places. Instead of *adding 0 to* f, they might equally well have thought of *multiplying f by 1*, finding some function h such that $h(x) = 1$ when $x =$ 1, 2, 3, and 4. Come up with some examples of that kind.

   Perhaps the simplest (non-trivial) one is $h(x) = (x-1)(x-2)(x-3)(x-4)+1$. There are infinitely many others, including $h(x) = x^2(x-1)(x-2)(x-3)(x-4)+1$ or $h(x) = (x^{100}+5)(x-1)(x-2)(x-3)(x-4)+1$.

5. Given a scatter plot like the one below, give two possible interpretations of what it means to find a function that "fits" the data?



   Depends on what we mean by "fits." If we define "fits" to mean that the function must contain each point (as students in the dialogue assumed) then we can come up with various functions that fit the data set. The interpretation used in statistics is different. A line (or other curve) of best fit does not need to pass through each point; instead, its goal is to provide a simple model that sufficiently approximates the data set.

Possible Responses to Related Mathematics Tasks

1. Find three different functions (of any kind) that fit through $(1,9)$, $(2,5)$, $(3,1)$, $(4,-3)$.

   Anything of the form:
   $13 - 4x + \underbrace{\text{anything}} \cdot (x-1)(x-2)(x-3)(x-4)$

2. Find three different functions (of any kind) that fit through $(1,1)$, $(2,3)$, $(3,6)$, $(4,10)$.

   Anything of the form:
   $\frac{1}{2}x^2 + \frac{1}{2}x + \underbrace{\text{anything}} \cdot (x-1)(x-2)(x-3)(x-4)$

3. Fill in the input-output table below for the function, $f(x)=\frac{5}{2}(x-1)(x-2)$.

x	$f(x)$
0	
1	
2	
3	

x	$f(x)$
0	0
1	0
2	0
3	5

4. A. Given the input-output table below, write a function that includes all the points.

x	$g(x)$
1	–7
2	0
3	0

We know $g(x)$ has roots at $x=$ 2 and 3. This means $g(x)$ has the factors $(x-2)$ and $(x-3)$. Using this we can write $g(x)=a(x-2)(x-3)$. Letting $x=1$ and $g(x)=-7$, we get $a=\frac{-7}{2}$ so $g(x)=\frac{-7}{2}(x-2)(x-3)$ is one function that agrees with the table.

B. Write a function for each of the input-output tables below.

x	$h(x)$
1	0
2	–1
3	0

x	$j(x)$
1	0
2	0
3	3

$h(x)=1(x-1)(x-3)$ and $j(x)=\frac{3}{2}(x-1)(x-2)$

C. Fill in the input-output table for $q(x) = g(x) + h(x) + j(x)$ given the input-output tables in parts A and B. Also write a function for $q(x)$ given the previous parts.

x	$q(x)$
1	$-7+0+0=-7$
2	$0+(-1)+0=-1$
3	$0+0+3=3$

$$q(x) = \frac{-7}{2}(x-2)(x-3) + 1(x-1)(x-3) + \frac{3}{2}(x-1)(x-2)$$

5. By first separating the table below into three input-output tables, write a function for $f(x)$.

x	$f(x)$
1	5
2	–2
3	8

Separating this input-output table into three parts produces the following tables:

x	$g(x)$
1	5
2	0
3	0

x	$h(x)$
1	0
2	–2
3	0

x	$j(x)$
1	0
2	0
3	8

Using the tables, the following functions can be written:

$$g(x) = \frac{5}{2}(x-2)(x-3)$$
$$h(x) = 2(x-1)(x-3)$$
$$j(x) = 4(x-1)(x-2)$$

Which means $f(x) = \frac{5}{2}(x-2)(x-3) + 2(x-1)(x-3) + 4(x-1)(x-2)$.