Dividing Fractions—Servings of Yogurt

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Dividing Fractions—Servings of Yogurt* Illustration: This Illustration's student dialogue shows the conversation among three students, with experience multiplying fractions and dividing whole numbers by fractions, trying to answer how many 3/4 cup servings of yogurt fit in 2/3 of a cup. They try several examples of dividing a whole number by a unit fraction (1/4) and then reason that if they are dividing by 3/4 instead of 1/4, the answer should be 1/3 the size. Next they try this same reasoning on examples where the dividend is a fraction, and find an answer to the original problem.

Highlighted Standard(s) for Mathematical Practice (MP)

MP 1: Make sense of problems and persevere in solving them.

MP 7: Look for and make use of structure.

MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grade 6

Target Content Domain: The Number System, Number & Operations—Fractions

Highlighted Standard(s) for Mathematical Content

6.NS.A.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?*

5.NF.B.7b Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.*

5.NF.A.2 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

Math Topic Keywords: fractions, division, multiplication, unit fractions

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This material is based on work supported by the National Science Foundation under Grant No. DRL-1119163. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

How many $\frac{3}{4}$-cup servings are there in $\frac{2}{3}$ of a cup of yogurt?

Task Source: Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). The Number System, 6–8.* Tucson, AZ: Institute for Mathematics Education, University of Arizona. http://commoncoretools.me/wp-content/uploads/2013/07/ccssm_progression_NS+Number_2013-07-09.pdf

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Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have already learned how to multiply two fractions and how to divide a whole number by a unit fraction and a unit fraction by a whole number. They are currently learning how to divide two fractions. They have been given a division-of-fractions problem with a context about serving size and have gotten to the point where they are trying to figure out what is $\frac{2}{3} \div \frac{3}{4}$. *They are using their understanding of unit fractions to figure out how to perform such a division.*

(1) Anita: We have to divide by $\frac{3}{4}$. Let's start by figuring out how to divide by *one* quarter. *[Pauses to think. Then, to Sam…]* Oh, this is going to be easy. How many quarters are in 1?

(2) Sam: 4.

(3) Anita: And 3?

(4) Sam: 12

(5) Anita: And 2?

(6) Sam: 8.

(7) Anita: And 5?

(8) Sam: 20.

(9) Dana: You're always just multiplying by 4.

(10) Anita: Right. And because $\frac{3}{4}$ is larger, there will be fewer $\frac{3}{4}$-size pieces in something than $\frac{1}{4}$-size pieces. Exactly a third as many. So how many *three*-fourths are there in 5?

(11) Sam: Umm, well…. There are 20 fourths in 5. Done. And 3 fourths is three times as big as 1 fourth so…. fewer three-fourths will fit in 5. A third of them in fact. A third of 20. That's how many three-fourths are in 5.

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(12) Dana: So to find how many three-fourths there are in any number, we multiply by 4 first to find out how many fourths there are and then we divide by 3 to find out how many three-fourths.

(13) Sam: Exactly, we multiply by 4 and divide by 3. Great! That should work for any number.

(14) Anita: Well, let's just do it once more. Let's try it with $7 \div \frac{3}{4}$. There are 28 fourths in 7; that's easy, just multiply by 4. And then we divide by 3 to find how many 3 fourths. $\frac{28}{3}$.

(15) Dana: How about another one, $4 \div \frac{3}{4}$. Multiply 4 by 4, that's 16. Divide by 3, that's $\frac{16}{3}$.

(16) Anita: That's a little bit more than 5. That sounds about right.

(17) Sam: Or $6 \div \frac{3}{4}$. Multiply by 4 is 24 and divide by 3 is 8.

(18) Dana: We can check it. 8 times 3 fourths is 24 fourths. That's 6, so it works.

(19) Anita: Or you can think of 3 quarters of 8 is 6.

(20) Dana: Wait a minute! To divide by $\frac{3}{4}$, we multiply by 4 and divide by 3. That's the same as multiplying by $\frac{4}{3}$.

(21) Sam: So can we go back to the original problem now?

(22) Dana: No, wait Sam! This is even more important than our silly problem! I think that method is going to work for all fractions! I don't even like yogurt.

(23) Sam: *What*'s going to work?

(24) Dana: To divide by $\frac{3}{4}$, we multiply by $\frac{4}{3}$. Look at those fractions. I'm sure it's going to work that way with all fractions. Let's try $\frac{2}{3} \div \frac{4}{7}$.

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(25) Sam: Eeeuw! Can't we work up to that with some more whole numbers?

[Again, they first check out how many sevenths in 1, in 2, in 5, and conclude they're always multiplying by 7. Then they check out how many four-seventh pieces in each of those, and decide it must be one-fourth as many as there were sevenths. A good quarter of an hour later…]

(26) Dana: It really does work! To divide by $\frac{4}{7}$, we multiply by 7 and then divide by 4, so we are multiplying by $\frac{7}{4}$. It really does work!!!

(27) Sam: Dana, you're way too excited! Take it easy!

(28) Dana: Don't you see? We invented a way to divide by any fraction, and we know why it works!!

(29) Sam: So, *now* can we go back to the original problem? How many $\frac{3}{4}$ cup servings are there in $\frac{2}{3}$ of a cup of yogurt? We said that's $\frac{2}{3} \div \frac{3}{4}$ and we needed to figure out how to divide those two fractions.

(30) Dana: Yes, and now we have a way to figure out how to divide by 3-fourths. Multiply by $\frac{4}{3}$.

(31) Anita: Well, two-thirds times *four* is *eight*-thirds. But we're not multiplying by 4, we're multiplying by $\frac{4}{3}$, so now we need to divide $\frac{8}{3}$ by 3. That's… 8-ninths.

(32) Sam: Why ninths?

(33) Anita: Because dividing by 3 is like taking a third of it. We want 1-third times 8-thirds. *[she writes* $\frac{1}{3} \times \frac{8}{3}$*]*. That's $\frac{8}{9}$.

(34) Sam: Oh right. We're just multiplying fractions. So if $\frac{2}{3} \div \frac{3}{4}$ is $\frac{8}{9}$, that means there are $\frac{8}{9}$ servings in $\frac{2}{3}$ cups of yogurt.

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(35) Dana: $\frac{8}{9}$ of a serving—not quite a full serving—sounds about right. A $\frac{3}{4}$-cup serving is more than the $\frac{2}{3}$ cup we have, but not by much. You would get just under a full $\frac{3}{4}$-cup serving from $\frac{2}{3}$ of a cup of yogurt. But why didn't you just use the simple method we found? To compute $\frac{2}{3} \div \frac{3}{4}$ we can simply do $\frac{2}{3} \times \frac{4}{3}$. That's easier.

(36) Sam: This is making me hungry. Even $\frac{3}{4}$ of a cup isn't a lot of yogurt, and we're only getting $\frac{8}{9}$ of that?! I'd want *more*!

(37) Dana: Eeeuw!

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

2. How do students use repeated calculations in lines 1–10 and later in lines 14–17? How does completing several calculations that all look similar help the students?

3. Think about the path students take in the Student Dialogue as they work through the problem. If you were overhearing this as it was taking place, are there places you would want to intervene? Places you would want to follow up on afterward? Ideas you would particularly want to share with the class, or alternatives you would want (later or immediately) to share with Anita, Dana, and Sam?

4. Based on the Student Dialogue, what issues might teachers want to keep in mind as they help engage students in MP 8: Look for and express regularity in repeated reasoning?

5. How did Anita multiply $\frac{2}{3}\times\frac{4}{3}$ in lines 31 and 33? What is the logic behind Anita's approach?

6. In line 35, Dana presents an argument to show that the result of their computation, $\frac{8}{9}$, is reasonable. What generalization can be made about when the result of a division computation is, as in this case, between 0 and 1? When can the result of a division be larger than *one* of the numbers that are being divided? When can the result of a division be larger than both of the numbers? Most importantly, which, if any, of these generalizations do you think are important for students to learn or make on their own?

7. Why are there two different notations—fractions and decimals—at all?! What are some of the advantages and disadvantages of representing each notation? As part of your answer, you might consider the following:

   A) Which feels easier to compute: $\frac{3}{7}\times\frac{5}{11}$ or the decimal equivalent $0.\overline{428571}\times0.\overline{454545}$?

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B) Which feels easier to compare in magnitude: $\frac{3}{7}$ and $\frac{5}{11}$ or their decimal equivalents $0.\overline{428571}$ and $0.\overline{454545}$?

C) Which feels easier to compute: $\frac{3}{10}+\frac{1}{2}$ or $0.3+0.5$?

D) Both $0.3333\ldots$ and $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\ldots$ mean the same thing. What are the advantages and disadvantages of each?

8. How can a visual model be used to represent $\frac{2}{3}\div\frac{3}{4}$ from the original task, which the students return to in line 29?

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Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Make sense of problems and persevere in solving them.	The Student Dialogue begins with Anita "looking for entry points to [the problem's] solution." Anita chooses a "simpler [form] of the original problem in order to gain insight into its solution" when she starts to divide by $\frac{1}{4}$ before considering the original task of dividing by $\frac{3}{4}$. In fact, Anita is strategically choosing to use an approach based on how many unit fractions (and later non-unit fractions) go into a quantity. Her approach does not rely upon any tools such as strip diagrams or number lines, but rather making sense about quantities and the relationship between quantities. This approach is apparent in lines 10 and 11 when the students reason about the difference in dividing by $\frac{3}{4}$ instead of $\frac{1}{4}$. Students continue thinking about units even when they "check their [answer]" in line 35 by recognizing that trying to put a bigger fraction (i.e., divisor) into the smaller fraction (i.e., dividend) will produce a quotient less than 1.
 Look for and make use of structure.	Students used the structure of fractions and quantities (lines 10–12) to extend division by $\frac{1}{4}$ to division by $\frac{3}{4}$. The realization that "exactly a third as many" $\frac{3}{4}$ will fit (line 10) is based on students' ability to see $\frac{3}{4}$ as $3\times\frac{1}{4}$. Throughout the Student Dialogue, students handle division by a fraction in two steps (e.g., lines 14, 17) based on treating non-unit fractions as integer multiples of a unit fraction. In line 20, Dana sees another aspect of the structure of the calculations they have been performing. Dana makes the connection that multiplying by 4 and dividing by 3 is the same as multiplying by $\frac{4}{3}$. The students then verify that dividing by a fraction is the same as multiplying by the reciprocal using other examples (lines 24–26), and they understand the significance of this structure as "a way to divide by any fraction" (line 28).

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Look for and express regularity in repeated reasoning.	MP 8 says that "mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts." However the students in this Student Dialogue move beyond *noticing* to actively *using* repeated calculations as a way to make sense of division by fractions. In lines 1–9, Anita has the group figure out how many one-quarters (or one-fourths) go into various numbers, and they come up with the shortcut of multiplying by 4. In lines 14–17, students use repeated reasoning again as they explore what it means to divide by three-fourths, leading them to realize that you can multiply by 4 and divide by 3.

Commentary on the Mathematics

Not surprisingly, Anita, Sam, and Dana approach the division of fractions in much the same way that they approached multiplication of fractions (see the Illustration *Multiplying Two Fractions*). Though they are figuring out a piece of middle school content, they are approaching it the way mathematicians might: trying to make sense of the problem they're faced with, and trying to *derive* a method that works, rather than looking up and following a formula. There is nothing wrong, of course, with learning and using an existing method—in fact, most of the time that is exactly what we must all do because there is not enough time to invent every technique afresh as if nobody had thought about it before. But it is *very* important that students also, and fairly regularly, have experiences *deriving* mathematical ideas and methods, for that is the way they learn to solve *new* problems—mathematical or otherwise—for which they do not have ready-made solution techniques. Even known problem types—word problems and proofs are common examples—often come in forms that don't make immediately obvious what mathematical idea or technique to use or what path to take, and students must be comfortable "playing" with them, perhaps taking various starting places and routes, until they find one that works.

Anita's route to figuring out how to divide by $\frac{3}{4}$ is to start with a simpler problem. Dividing by $\frac{1}{4}$, she says, is easy and she makes use of the idea that dividing by something bigger should give a result that is smaller. More precisely, $n \div \frac{3}{4}$ should give a result that is one-third the size of $n \div \frac{1}{4}$. In *measurement*, this idea was developed very early in elementary school (in grade 2 in CCSSM); here we see it applied in a sophisticated and more precise way in the context of arithmetic (*not* measurement) in middle school.

Anita establishes, by repeated calculations (lines 1–9), that to compute the quotient $n \div \frac{1}{4}$, we may instead compute the product $n \times 4$. There is no formality in this approach, but nobody needs the formality: the structure of this calculation becomes obvious from the examples. If Anita's friends had needed to start a bit further back, Anita might first have had to establish that "n divided by d" can be interpreted as meaning "how many d in n?" and the examples could be

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whole numbers: how many 2s in 10, in 6, in 14? And *then* how many *halves* in 2, 1, 5, 3, 8, 1, $1\frac{1}{2}$, etc.? And *then* how many quarters in…?

After establishing that $n \div \frac{1}{4} = n \times 4$, Sam and Dana take quickly to the idea that $n \div \frac{3}{4}$ is one-third of $n \div \frac{1}{4}$ and, therefore, $n \div \frac{3}{4} = n \times 4 \div 3$. Again, a different group might not take that step so quickly, and might need to work through a few other concrete examples (MP 8).

It is Dana (line 20), who notices the structure of this calculation. Dana sees $n \times 4 \div 3$ as $n \times \frac{4}{3}$. Dana is already completely sure of this observation, having noticed how the *arithmetic* of $n \div \frac{a}{b}$ will depend on the values of a and b, but the *structure* of the computation is always the same: multiply by the bottom number to see how many $\frac{1}{b}$ there are, and divide by the top number to scale the result. Even so, Dana checks it out concretely just to go through the process once more to ensure that the generic form has been correctly abstracted. Together, Anita, Sam, and Dana have "invented" division of fractions—not, of course, for the first time, but for themselves. If someone now says "invert and multiply," they will be able to explain why that works!

Evidence of the Content Standards

The students in this Student Dialogue are trying to figure out how to divide two fractions in the context of a word problem about serving size (6.NS.A.1). In their work trying to figure out how to divide fractions, students begin by looking at simpler cases of dividing whole numbers by a unit fraction (5.NF.B.7b). Students are also flexibly interpreting fractions as a division of the numerator by the denominator (and vice versa) (5.NF.A.2). This is seen in Anita's approximation of $\frac{16}{3}$ as being slightly more than 5 (line 16) and in Dana's conversion of 24 fourths to 6 (line 18). Moving from division to fraction notation is also seen in line 20 when Dana converts "multiply by 4 and divide by 3" to "multiplying by $\frac{4}{3}$."

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. Before even starting to calculate exactly, approximately how many $\frac{3}{4}$-cup servings are there in $\frac{2}{3}$ of a cup of yogurt? One serving? Two? Five? Between some of these? Less than one?

2. Explain in your own words the reasoning that the students use in the Student Dialogue to claim that $\frac{2}{3} \div \frac{3}{4}$ and $\frac{2}{3} \times \frac{4}{3}$ give the same result.

3. In line 16, Anita says that $\frac{16}{3}$ is a little more than 5, which "sounds about right." Why does it sound about right?

Related Mathematics Tasks

1. When you divide a number by something, is the result (called the quotient) always a smaller number? Explain your answer.

2. How many $\frac{1}{3}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{1}{3}$ that go into a number.

3. How many $\frac{2}{3}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{2}{3}$ that go into a number.

4. How many $\frac{1}{b}$ are in 1, 2, 5, 10, 100, n?

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5. How many $\frac{a}{b}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{a}{b}$ that go into a number.

6. Estimate without calculating whether the following quotients are less than, equal to, or more than 1. Explain how you came up with your estimates.
   A) $\frac{1}{3} \div \frac{1}{2}$
   B) $\frac{9}{10} \div \frac{5}{6}$
   C) $\frac{4}{5} \div \frac{8}{10}$
   D) $\frac{4}{3} \div \frac{6}{5}$

7. Oma is baking apple strudel for a family reunion. She has $5\frac{1}{2}$ cups of flour and each strudel requires $1\frac{1}{3}$ cups of flour. How many strudels can she bake?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. How do students use repeated calculations in lines 1–10 and later in lines 14–17? How does completing several calculations that all look similar help the students?

   Students repeat several calculations of "how many times does $\frac{1}{4}$ go into x" in lines 1–10 in order to see a pattern and state the shortcut that you just multiply by 4. Later in lines 14–17, students repeat several calculations of "how many times does $\frac{3}{4}$ go into x" to discover that they are always multiplying by 4 and dividing by 3. Using repeated calculations helps students see and express regularity.

3. Think about the path students take in the Student Dialogue as they work through the problem. If you were overhearing this as it was taking place, are there places you would want to intervene? Places you would want to follow up on afterward? Ideas you would particularly want to share with the class, or alternatives you would want (later or immediately) to share with Anita, Dana, and Sam?

   This question is completely open to discussion. There is no "right" answer for all students and teachers. But, as a list of potential issues, one might consider the following:
   - Is there an optimal amount of time for students to repeat a process in order to see the regularity in it? How do you decide when enough is enough? Does that depend on the process? The students? Your goals? The schedule?
   - *These* 6th graders happen to be thinking in a very particular way, making use of an MP 7 search for and use of structure, and taking an MP 8 approach to conducting that search. It might well be faster or clearer to get them to use an area model, or just to teach them the standard algorithm after some suitable preparation for understanding it (like examples, with whole numbers, of multiplying by the reciprocal). How can you decide whether you want to let them pursue this path or move them to another?

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4. Based on the Student Dialogue, what issues might teachers want to keep in mind as they help engage students in MP 8: Look for and express regularity in repeated reasoning?

   MP 8 involves three steps. The first is engagement in some type of repeated reasoning (e.g., a calculation, argument, geometric construction). The second is looking for some type of regularity or common element that ties all of these repeated cases together. And finally, students find some way of expressing that regularity.

   Looking at the Student Dialogue, we see that students take their time repeating division by $\frac{1}{4}$ (lines 1–9) and then repeating division by $\frac{3}{4}$ (lines 10–19). They did this in order to find and express regularity; whether this is multiplying by 4 when dividing by $\frac{1}{4}$ (line 9) or multiplying by $\frac{4}{3}$ when dividing by $\frac{3}{4}$ (line 20). Even after students have found and expressed the regularity in their repeated reasoning, they go back to trying more repeated examples (lines 24–26) to convince themselves that their "multiply by the reciprocal" shortcut works. The way students behave in this Student Dialogue reflects the different paces at which students move and shows how, even after expressing the regularity, they may need to fall back on a few more examples to convince themselves of their shortcut. It is also worth keeping in mind that students may find the regularity in repeated reasoning without necessarily being able to express it in some abstract way. In the Student Dialogue, students find the regularity when dividing by $\frac{3}{4}$ for some time (line 13), but not until line 20 can they express that regularity as multiplying by $\frac{4}{3}$. While it is important for teachers to give students time to engage in repeated reasoning, it is also important to encourage them to look for the regularity. Doing many repeated calculations without trying to find a pattern or shortcut is *not* MP 8. In general, it also does students little good.

5. How did Anita multiply $\frac{2}{3}\times\frac{4}{3}$ in lines 31 and 33? What is the logic behind Anita's approach?

   In lines 31 and 33, Anita does not use the standard algorithm, multiplying the numerators and denominators. Instead, Anita identifies that $\frac{2}{3}$ is 2 *thirds* (two of some object or unit); when we multiply that by 4 (from 4 thirds) we get 8 of that object or unit, 8 thirds. She then corrects: we want to multiply by 4 *thirds*, not just 4, so we need to take a third of the result we got when we multiplied by 4. One-third of 8 thirds is 8 ninths. She doesn't keep the same unit. If she were *really* thinking of eight-thirds as if it meant 8 objects (e.g., 8 candy bars), she would have concluded that the final answer is $2\frac{2}{3}$ thirds

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(one-third of the 8 "candy bars") and might even have written $\frac{2\frac{2}{3}}{3}$. This rather odd-looking number is quite correct, but certainly not familiar. Instead, Anita took one-third of *each* "third" in the 8 thirds, giving 8 ninths. Anita's explanation is an example of what it means to reason through a computation instead of following an algorithm. While an algorithm is helpful in making calculations more efficient, it is important to be able to understand why it works as well. To explore multiplying fractions further, see the Illustration *Multiplying Two Fractions*.

6. In line 35, Dana presents an argument to show that the result of their computation, $\frac{8}{9}$, is reasonable. What generalization can be made about when the result of a division computation is, as in this case, between 0 and 1? When can the result of a division be larger than *one* of the numbers that are being divided? When can the result of a division be larger than both of the numbers? Most importantly, which, if any, of these generalizations do you think are important for students to learn or make on their own?

   A division computation will be between 0 and 1 when a divisor is larger than a dividend (both need to be positive or both need to be negative). Since the divisor can't fit into the dividend one full time, the quotient will be a number between 0 and 1. Similarly, a division will result in a number larger than 1 when the dividend is larger than the divisor (and either both are positive or both are negative).

   The result of a division, $\frac{a}{b}$, will be larger than both a and b when one of several cases is met:

   - If both a and b are negative, which results in a positive value for $\frac{a}{b}$.
   - If the dividend is bigger than 1 and the divisor is between 0 and 1 (i.e., $a > 1$ and $0 < b < 1$).
   - If the dividend is a positive fraction less than 1 and the divisor is a positive fraction less than that (i.e., $0 < a < 1$ and $0 < b < a$).
   - If the dividend is a positive fraction less than 1 and the divisor is both larger than the dividend but smaller than the square root of the dividend (i.e., $0 < a < b < 1$ and $b < \sqrt{a}$). This happens since $a < \sqrt{a}$ when $0 < a < 1$, and if $b < \sqrt{a}$, then that guarantees that (1) $b < \frac{a}{b}$ and (2) $\frac{a}{b} > \sqrt{a}$, making $\frac{a}{b} > a$ as well.

   Choosing which of the generalizations made above you want students to learn or make on their own depends on you and the needs of your students. While some generalizations such as when a division will result in an answer less than or greater than 1 might be something you want all students to make on their own, some students may need more support to make other generalizations.

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7. Why are there two different notations—fractions and decimals—at all?! What are some of the advantages and disadvantages of representing each notation? As part of your answer, you might consider the following:

   A) Which feels easier to compute: $\frac{3}{7}\times\frac{5}{11}$ or the decimal equivalent $0.\overline{428571}\times 0.\overline{454545}$?

   B) Which feels easier to compare in magnitude: $\frac{3}{7}$ and $\frac{5}{11}$ or their decimal equivalents $0.\overline{428571}$ and $0.\overline{454545}$?

   C) Which feels easier to compute: $\frac{3}{10}+\frac{1}{2}$ or $0.3+0.5$?

   D) Both 0.3333… and $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\ldots$ mean the same thing. What are the advantages and disadvantages of each?

In general, it is far easier to *compute* with fractions than with decimals, though if the decimals terminate (or if we lop them off arbitrarily at some level of precision because we don't *need* complete precision), they can sometimes be easier to compute with. In case A, it's clear that computing with the infinite decimals—or even finite approximations of them such as the six-place representations *without* infinite repetition—is unreasonably difficult, while representing the same numbers as fractions makes the calculation straightforward.

By contrast, it is generally far easier to *compare* magnitudes of numbers represented as decimals than as fractions, though, again, there are exceptions. Using decimals allows us to compare place value, digit-by-digit, as in case B. When fractions happen to have the same numerator or the same denominator, they are as easy to compare as decimals. For example, if the fraction-representation of the numbers in case B happened to be $\frac{21}{77}$ and $\frac{35}{77}$, the comparison would be trivial. To explore different ways one might compare two fractions, see the Illustration *Comparing Fractions*.

In C, the (finite) decimal calculation is easier.

And, importantly, fractions give information about structure and relationships—family lineage might be a good analogy—that decimals do not. Students, even in 6^{th} grade, will understand certain translations between decimal and fraction form, recognizing, for example 0.33 as $\frac{33}{100}$. It is less likely that they would recognize

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0.33 as $\frac{3}{10}+\frac{3}{100}$, let alone the very familiar 0.25 as $\frac{2}{10}+\frac{5}{100}$, but both of those are still consistent with their learning and are potentially interesting relationships to check out. In even those finite cases, let alone $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\ldots$, the decimal is both simpler to write than the fraction sums and easier to read and interpret. And, because the denominators in these examples are powers of 10, there is no point in *not* using the decimal notation, which is precisely a shorthand for such sums (just as base 10 integer notation is a shorthand for sums like 6 × 100 + 4 × 10 + 7). But the attempt to "recognize" an only slightly different example—$\frac{3}{8}+\frac{3}{8^2}+\frac{3}{8^3}+\frac{3}{8^4}+\frac{3}{8^5}+\cdots$—illustrates another way in which fractions and decimals call attention to different "traits" of a number, and may also help clarify *when* one notation is more useful than the other. If one is patient enough, and carries the computation far enough, one can experimentally see that $\frac{3}{8}+\frac{3}{8^2}+\frac{3}{8^3}+\frac{3}{8^4}+\frac{3}{8^5}+\cdots$ gets close and stays close to 0.4285…. In decimal notation, a reasonable question is "so what?" We figured it would have *some* value, so why not *that* one? No insight is gained. But if we conclude, either from the way we compute or from having a hunch about that decimal, that this sum is getting close to $\frac{3}{7}$, we begin to think harder about why it is *that* number and not some other. The $\frac{3}{7}$ suggests family relationship, *structure*, in a way that an arbitrary decimal does not.

8. How can a visual model be used to represent $\frac{2}{3}\div\frac{3}{4}$ from the original task, which the students return to in line 29?

Students can think about division of fractions as "what times the divisor gives the dividend?" (i.e., rewriting $\frac{2}{3}\div\frac{3}{4}$ as $?\times\frac{3}{4}=\frac{2}{3}$). One visual model that can help is tape diagrams (or the equivalent on a number line). Using two tape diagrams (each representing 1), represent $\frac{2}{3}$ the dividend (top bar in the image below) and $\frac{3}{4}$ the divisor (bottom bar).

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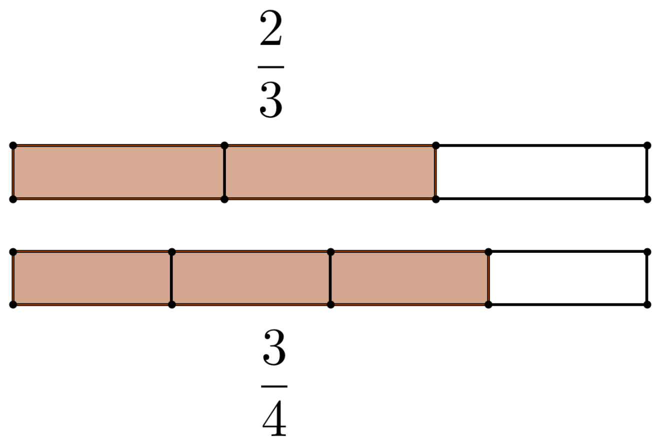
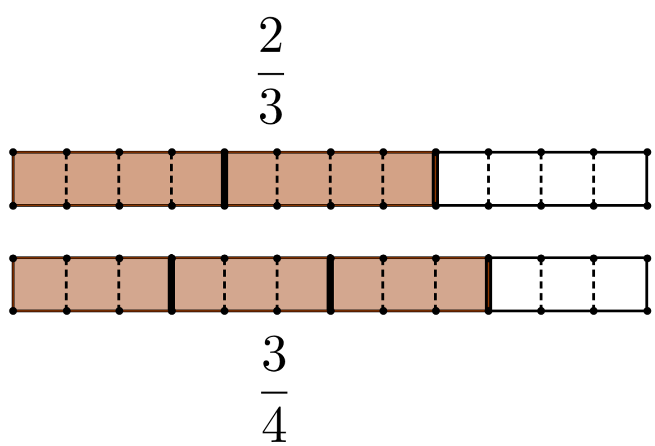


Then subdivide the two bars into smaller partitions that "go evenly" into both thirds and fourths (the current partitions)—that is, *integer* multiples of these smaller bits will make both thirds and fourths—as seen in the picture below.



The dividend contains 8 of these smaller partitions and the divisor contains 9 of them. The quotient is the number of times the divisor (bottom shaded bar) fits into the dividend (top shaded bar). So the top shaded area is eight-ninths of the bottom shaded area. This means $\frac{8}{9}\times\frac{3}{4}$ is equal to $\frac{2}{3}$. And rewriting $\frac{8}{9}\times\frac{3}{4}=\frac{2}{3}$, we get $\frac{2}{3}\div\frac{3}{4}=\frac{8}{9}$.

Dividing Fractions—Servings of Yogurt

Possible Responses to Student Discussion Questions

1. Before even starting to calculate exactly, approximately how many $\frac{3}{4}$-cup servings are there in $\frac{2}{3}$ of a cup of yogurt? One serving? Two? Five? Between some of these? Less than one?

   We would expect there to be a little less than 1 serving because we only have $\frac{2}{3}$ cups of yogurt and this is less than a full serving, which is $\frac{3}{4}$ cups.

2. Explain in your own words the reasoning that the students use in the Student Dialogue to claim that $\frac{2}{3} \div \frac{3}{4}$ and $\frac{2}{3} \times \frac{4}{3}$ give the same result.

   Let students summarize their understanding of the Student Dialogue. In their response, they should discuss the strategy used by students in the Student Dialogue in reasoning about how many quarters (or fourths) fit into various numbers before moving on to figuring out how many **three**-fourths fit into different numbers. This eventually leads to an observation (line 20) that was then verified with other fractions (lines 24–26) and turned into a generalization about dividing by fractions (line 28).

3. In line 16, Anita says that $\frac{16}{3}$ is a little more than 5, which "sounds about right." Why does it sound about right?

   $\frac{16}{3}$ is a little more than 5 because $\frac{15}{3}$ equals 5. This "sounds about right" because the calculation (line 15) is $4 \div \frac{3}{4}$ and, since $\frac{3}{4}$ is less than 1, it will go into 4 more than 4 times.

Possible Responses to Related Mathematics Tasks

1. When you divide a number by something, is the result (called the quotient) always a smaller number? Explain your answer.

   Division can result in a quotient that is smaller than the dividend or larger than the dividend. For example, if we divide a whole number by a smaller whole number, the quotient will be smaller than the dividend (as in 10 divided by 2). If, however, we divide a whole number by a unit fraction (like in lines 1–8) or by a non-unit fraction less than 1 (like in line 17), we get a quotient that is larger than the dividend. The reason why you get a larger quotient has to do with how many times the divisor can fit into the whole

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number dividend. If the divisor is 1, for example, it will fit into the dividend number the same number of times as the dividend itself. However, if the divisor is smaller than 1, it will be able to fit into the dividend a greater number of times. Using a number line one sees this with examples. For instance, the number of halves, quarters, etc., that fit into a whole number will be greater than the number of 1s that will fit into the same whole number.

2. How many $\frac{1}{3}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{1}{3}$ that go into a number.

   There are three $\frac{1}{3}$ in 1. There are 6 in 2. There are 15 in 5. There are 30 in 10. There are 300 in 100. There are $3n$ in n. To find the number of $\frac{1}{3}$ that go into a number, we multiply the number by 3.

3. How many $\frac{2}{3}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{2}{3}$ that go into a number.

   There are three $\frac{1}{3}$ in 1; there are half as many $\frac{2}{3}$ in 1 so instead of 3 we get $\frac{3}{2}$. There are 6 one-thirds in 2 and half as many two-thirds in 2, which means there are $\frac{6}{2}$ (or 3) two-thirds in 2. There are $\frac{15}{2}$ two-thirds in 5. There are $\frac{30}{2}$ (or 15) two-thirds in 10. There are $\frac{300}{2}$ (or 150) two-thirds in 100. There are $\frac{3n}{2}$ two-thirds in n. To find how many $\frac{2}{3}$ are in a number, we first multiply the number by 3 (to get the number of $\frac{1}{3}$ in the number) and then we divide by 2 (to get the number of $\frac{2}{3}$).

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4. How many $\frac{1}{b}$ are in 1, 2, 5, 10, 100, n?

   $b \times \frac{1}{b}$ is 1. $2b \times \frac{1}{b}$ is 2.
   $5b \times \frac{1}{b}$ is 5.
   There are $10b$ in 10.
   There are bn in n.

5. How many $\frac{a}{b}$ are in 1, 2, 5, 10, 100, n? Describe in words the process you use to find the number of $\frac{a}{b}$ that go into a number.

   We already know how to find out how many $\frac{1}{b}$s in any number. $\frac{a}{b}$ is bigger than $\frac{1}{b}$ by a factor of *a*, so there will be fewer $\frac{a}{b}$ in that number by exactly the factor *a*. To find out how many $\frac{a}{b}$ in *n*, we multiply *n* by b and then divide by a.

6. Estimate without calculating whether the following quotients are less than, equal to, or more than 1. Explain how you came up with your estimates.
   A) $\frac{1}{3} \div \frac{1}{2}$
   B) $\frac{9}{10} \div \frac{5}{6}$
   C) $\frac{4}{5} \div \frac{8}{10}$
   D) $\frac{4}{3} \div \frac{6}{5}$

   A) $\frac{1}{3} \div \frac{1}{2}$ means "how many halves in a third." Less than 1, because $\frac{1}{2}$ is bigger than $\frac{1}{3}$.
   B) $\frac{9}{10} \div \frac{5}{6}$ will be greater than 1 because $\frac{5}{6}$ is less than $\frac{9}{10}$. To see that $\frac{5}{6}$ is in fact smaller than $\frac{9}{10}$, we can think "$\frac{5}{6}$ is a sixth less than 1 and $\frac{9}{10}$ is 1 tenth less than 1, which makes it *closer* to 1 because tenths are smaller than sixths."

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C) $\frac{4}{5} \div \frac{8}{10}$ will be 1 because $\frac{4}{5} = \frac{8}{10}$.

D) $\frac{4}{3} \div \frac{6}{5}$ is greater than 1 because $\frac{6}{5} < \frac{4}{3}$. How can we know that easily? We know that $\frac{4}{3}$ is $\frac{1}{3}$ more than 1 and $\frac{6}{5}$ is $\frac{1}{5}$ more than 1. Because $\frac{1}{5}$ is smaller than $\frac{1}{3}$ that means $\frac{6}{5}$ adds less to 1 than $\frac{4}{3}$, so it's smaller.

7. Oma is baking apple strudel for a family reunion. She has $5\frac{1}{2}$ cups of flour and each strudel requires $1\frac{1}{3}$ cups of flour. How many strudels can she bake?

To calculate the number of strudels, we need to divide the cups of flour available by the cups of flour needed for each strudel: $5\frac{1}{2} \div 1\frac{1}{3}$. This can be rewritten as $\frac{11}{2} \div \frac{4}{3}$ and can be calculated by multiplying $\frac{11}{2}$ by 3 and then dividing that result by 4. This can be written as $\frac{11}{2} \times \frac{3}{4} = \frac{33}{8}$. The result, $\frac{33}{8}$, is just a little over 4 (exactly $4\frac{1}{8}$), which means Oma can bake 4 strudels.

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