Finding Parallelogram Vertices

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Finding Parallelogram Vertices* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the fourth vertex of a parallelogram given only three vertices. While students use the properties of parallelograms and an informal understanding of slope to find a fourth vertex, they eventually come to realize more than one point could be the fourth vertex.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 5–7

Target Content Domain: Geometry

Highlighted Standard(s) for Mathematical Content

5.G.B.4	Classify two-dimensional figures in a hierarchy based on properties.
6.G.A.3	Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
7.G.A.2	Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Math Topic Keywords: properties of parallelograms, properties of quadrilaterals, coordinate plane

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Given three vertices of a parallelogram at $A(1,2)$, $B(4,1)$, and $C(5,3)$, where can the fourth vertex be located?

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Student Dialogue

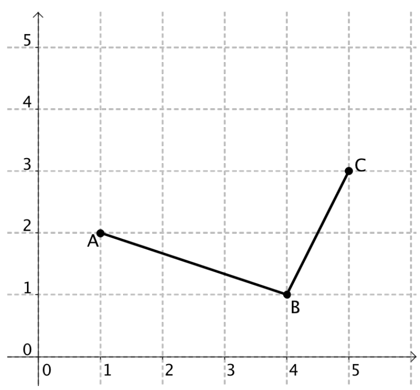
Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students in this dialogue have just studied the properties and definition of different polygons. They are now working on open-ended problems that require the use of those definitions and properties. Students already have experience plotting points onto the Cartesian plane from previous units.

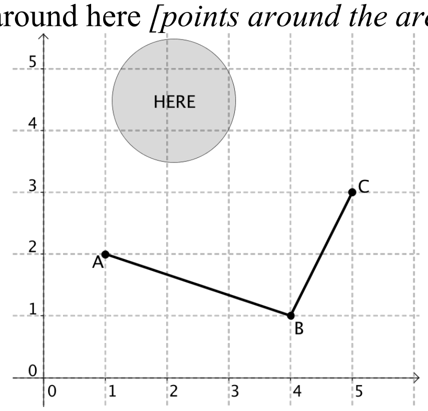
(1) Sam: Well, let's use graph paper and plot the three points we know.
[plots on graph paper]
Now all we need to do is find one more point that will make a parallelogram. What do we know about parallelograms?

(2) Anita: Opposite sides are parallel?

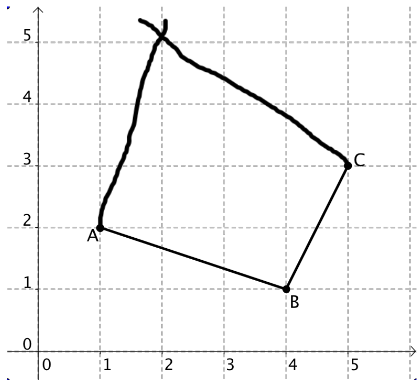
(3) Dana: So, let's draw some sides and see if we can't find the fourth vertex. We can connect points A and B to get side AB, and we can connect points B and C to get side BC.



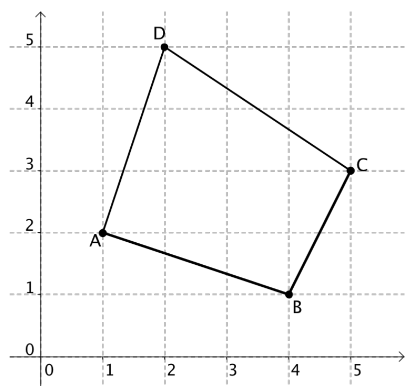
(4) Sam: This is starting to look more like a parallelogram! Hmm… it looks like the fourth point should be around here *[points around the area indicated by the gray circle]*.



Here. What if I kind of draw the lines like this… *[sketches two shaky lines, one from A and C]*



(5) Dana: It looks like the lines meet at (2,5). But I can't tell if your lines are quite parallel. It doesn't look like it. Here, let me draw it with a ruler. *[erases the sketched lines, plots the point (2,5), and connects the remaining sides with a ruler]*



(6) Anita: That doesn't look parallel at all! Definitely not a parallelogram!

(7) Sam: Yeah, that doesn't look right. Let's try again. But wait! Even if the picture *looks* right, does that mean it's really a parallelogram? And what if the fourth point is not even on a gridline?

(8) Anita: I think there's a way to find the point we want without guessing. If we picture side *BC* sliding over, we would get the opposite side of the parallelogram. *[uses hand to show how the line segment would slide]*

(9) Dana: You're keeping your hand pointed in the same direction as you move it—does that help us?

(10) Anita: Yeah, I'm trying to keep the line tilted the same way so they're parallel. And we can do the same thing to slide *AB* up. But it's still hard to see where exactly the fourth point would end up.

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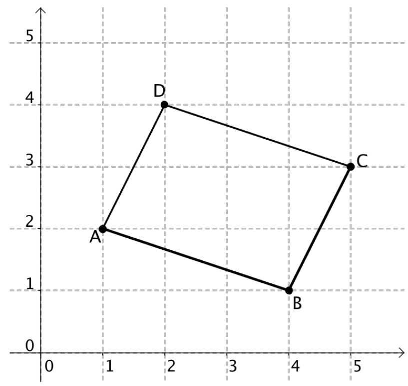
(11) Sam: We have the graph paper, so why don't we use the grid? To move from *B* to *C*, we have to go up 2 and over 1. So let's go up 2 and over 1 from *A* and see where that gets us. That will make sure the side is the same length and tilted the same way. *[students go up 2 and over 1 and get (2,4) and mark the new point* D*]*



(12) Anita: They're tilted the same way, so we know that *AD* is parallel to *BC*. And they have to be the same length because they cross the same amount of space. Let's check if *CD* is parallel to *BA*. *[students use the same reasoning to check the remaining two sides of the parallelogram]*

(13) Sam: Great! We're done!

(14) Dana: Wait. We might not be done. I'm wondering if we can have more parallelograms.

(15) Sam: What do you mean?

(16) Dana: Well, I originally connected points *A* and *B* to get side *AB* and points *B* and *C* to get side *BC*, but I could've drawn *AB* and *AC* instead, right?

(17) Anita: You're right! We've only considered one possible parallelogram but there could be more. Let's find those too…

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. Find the coordinates of all other possible fourth vertices from the task used in the dialogue.

3. How do you know you have found all the possible fourth vertices?

4. What if the students had said there aren't any other possible vertices aside from (2,4)? How would you help students see that other vertices are possible?

5. How could a high school student with knowledge of algebra solve this problem?

6. What did using graph paper as opposed to blank paper allow the students to do? How/why can graph paper be considered a tool? Go back to the dialogue to find evidence of how using graph paper contributed to the students' thinking as they worked to solve the problem.

7. What might be the affordances and the risks in providing Geoboards to the students in the dialogue?

8. If this dialogue were happening as a classroom discussion, where in the dialogue would you ask students to repeat their thinking for extra emphasis, ask for more clarification, or otherwise draw the class's attention to students engaging in the mathematical practices?

9. Given points (2,1), (6,1) and (2,-2), how many parallelograms can you make? How many rectangles? Explain your reasoning.

10. If the points $A(0,0)$, $B(5,0)$, and $C(8,3)$ were the vertices of an isosceles trapezoid, how many possible fourth vertices could you have? Justify your answer.

11. If the points $A(0,0)$, $B(5,0)$, and $C(8,3)$ were the vertices of a trapezoid (not necessarily isosceles), how many possible fourth vertices could you have? Justify your answer.

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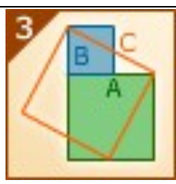
Mathematical Overview

Suggested Use The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

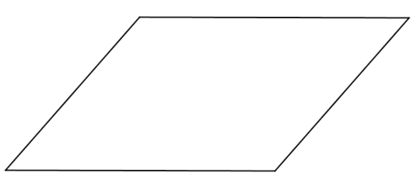
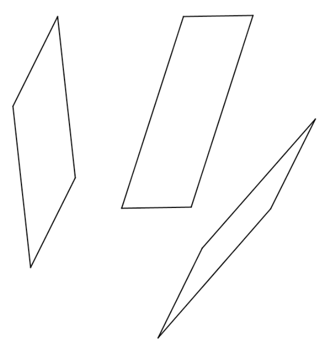
Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	Many students experience mathematics as if it is all about following prescribed methods, but this problem has no such pre-determined route to the solution. The students must make sense of the problem and persevere in solving it. They plan out their own response, choose graph paper as an appropriate tool (line 1; MP 5), and "monitor and evaluate their progress and change course" (MP 1), refining their initial guess and checking their process by thinking about properties of parallelograms (lines 2, 8). They "understand the approaches of others… and identify correspondences between different approaches" (MP 1) when Sam suggests counting using the graph paper in response to Anita's approach of preserving tilt using hand motions (lines 8–12). Students also think about the assumptions they make and consider other possible cases (lines 14–17).
 Construct viable arguments and critique the reasoning of others.	Students use the properties of parallelograms to construct their argument (line 2). They build an argument together as they recognize the need to become more precise in justifying the location of the fourth point. The students start by drawing and sketching, but Sam points out a potential flaw in their reasoning by questioning the validity of determining a parallelogram by what "looks right" (line 7). The students "justify their conclusions, communicate them to others, and respond to the arguments of others" (MP 3) when Sam adds a numerical observation to Anita's visual one (lines 8–11), and Anita's response includes justification for Sam's contribution (line 12). Finally, at the end of the dialogue, the students have to check their assumptions when Dana realizes they might not be done and the students must consider other possible cases (lines 13–17).
 Use appropriate tools stratgeically.	The students use graph paper and a ruler to help them solve the problem. However, it is not simply their use of these tools that demonstrates the thinking highlighted by this MP. The students start by plotting points, and then Sam suggests sketching lines to estimate a possible fourth point (line 4). Dana responds by suggesting they use a ruler to help determine whether Sam has really found the fourth point (line 5). The students recognize that drawing the figure using a straightedge is useful, but Sam also realizes a limitation of the tool by arguing that a picture that "looks right" doesn't necessarily guarantee a parallelogram (line 7). Then the

	students use an intuitive precursor to the notion of slope when they count the units up and over from point *B* to *C* to determine the length and slant of $\overline{BC}$ and then reproduce that same length and slant starting at point *A* to get the final vertex (line 11). For middle school students, this is an inventive and appropriate use of the coordinate plane to *match* a distance, even when they are not yet able to calculate that distance precisely, and the students demonstrate they recognize the "insight to be gained" (MP 5) from using graph paper not just to locate points, but to quantify the relationship between points.
 Look for and make use of structure.	Students look for and make use of structure. This can be seen when the students use the properties of parallelograms, namely that opposite sides are parallel (and congruent). These properties are used in lines 8–12 when the students are sliding over one side to get the opposite side of the parallelogram, attending to the structure formed by the relationships between the given points and the point they are trying to find. In this process, students are using structure when they see that counting "up and over" is a reliable way to produce a segment with the same "tilt" and the same length and, therefore, is a way to determine the fourth vertex of the parallelogram precisely.

Commentary on the Mathematics

<u>Common misconception: Overreliance of idealized "standard" shapes</u>
This dialogue also serves to illustrate a common tendency to over-rely on idealized images of shapes. Given the points (1,2), (4,1), and (5,3), a natural first tendency is to look for the final vertex in the area of (2,4). Parallelograms are very often depicted like the example on the left, below, and less often shown in general shapes and positions.

 Idealized "standard" image of parallelogram	 Generic parallelograms in general positions

Limiting the examples—in this case, examples of parallelograms—that students see narrows the image they generate and can lead students to depend on the idealized image rather than the concept definition (necessary and sufficient properties). In the problem students explore in this dialogue, idealized images about the shape and orientation of parallelograms make (0,0) and

(8,2) less visible and "tempting" choices. The use of "standard" idealizations in geometry is problematic with other shapes such as triangles and trapezoids as well. When called upon to think of or sketch or state properties of a triangle, students whose image is isosceles or equilateral are likely to overgeneralize properties and relationships that don't occur in all cases.

Extensions to upper grades

While this dialogue is targeted to middle school content, the math problem posed would be equally appropriate for Algebra I students, who learn to calculate slope, learn that parallel lines have equal slopes, and might apply that notion to find and check solutions. The problem in this dialogue can be extended to high school geometry students as well. Once students find all three possible locations for the fourth vertex, they will notice that those fourth vertices form a triangle whose sides have midpoints that are the original three vertices of the parallelogram. A rigorous proof of this observation can be done and is the subject of the Illustration titled *Proof with Parallelogram Vertices*.

Evidence of the Content Standards

Students in the dialogue draw geometric shapes on the coordinate plane (e.g., lines 3–5, line 11) in an attempt to fulfill the parallelogram condition (7.G.A.2 and 6.G.A.3). Students are able identify and use properties of parallelograms (5.G.B.4) such as opposite sides are parallel (line 2) and congruent (lines 11 and 12). In fact these properties of parallelograms, along with the aid of the coordinate system's grid, allow students to "measure" the length of parallelogram sides using an "up and over" technique (line 11) (6.G.A.3).

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. What does Sam do in line 11 of the dialogue to "measure" the side of the parallelogram? Will this method always work to measure the side of a parallelogram?

2. In line 11, why does Sam move up 2 units and over 1 unit from point *A*?

3. In line 2, Anita identifies a property of parallelograms that the students use to find the fourth vertex in the dialogue. What is the property? What are other properties of parallelograms that the students could also use?

4. What assumptions do the students make in the beginning of their problem-solving process that Dana discovers in lines 14 and 16? How does making the assumptions affect their problem-solving process?

Related Mathematics Tasks

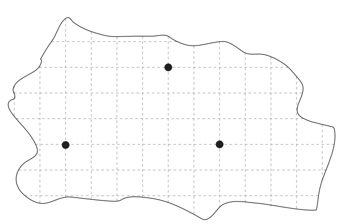
1. Find the coordinates of all other possible fourth vertices of the parallelogram described in the problem.

2. How do you know you have found all the possible fourth vertices? Justify your answer.

3. Given points (2,1), (6,1), and (2,-2), how many parallelograms can you make? How many rectangles? Justify your answer.

4. Name three points that cannot be the vertices of a parallelogram, no matter where the fourth point is. What must be true about these three points?

5. Four keys are required to open a treasure vault. Your ripped map shows the location of three of the keys. You know the four keys are located at the corners of a parallelogram. Where could the fourth key be?



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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. Find the coordinates of all other possible fourth vertices from the task used in the dialogue.

   The possible fourth vertices are (2,4), (0,0), and (8,2).

3. How do you know you have found all the possible fourth vertices?

   Initially, we are only given three vertices of the parallelogram. Like in the dialogue, we can connect the vertices to start drawing the parallelogram. Considering all the different possible scenarios—we can connect the vertices to get $\overline{AB}$ and $\overline{BC}$ (case #1), $\overline{AC}$ and $\overline{BC}$ (case #2) or $\overline{AB}$ and $\overline{AC}$ (case #3). Another way to justify why there are only three possible points for the fourth vertex is to think about opposite corners of a parallelogram. The fourth vertex of the parallelogram must be opposite to some other vertex. Since we have three vertices to choose from that must mean the fourth vertex may be in one of three places.

   Please note: Asking questions such as "how do you know you're done" or "justify your answer" encourages engagement in MP 3: Construct viable arguments and critique the reasoning of others. Did you engage in that MP when answering the question?

4. What if the students had said there aren't any other possible vertices aside from (2,4)? How would you help students see that other vertices are possible?

   Students could describe in detail the process they used to get the vertex (perhaps even writing down the steps explicitly). Students may have forgotten that the original task gave three points, not two lines. This is a chance for students to engage in MP 1 as they make sure they have made proper sense of the problem. You may also ask students to look for assumptions they made. Have the students ask themselves, "Why did I choose to do the step this way, and is there another mathematically correct alternative to the way I did it?" Identifying assumptions we make in our thinking is important since that allows us to consider new possibilities in a problem or solutions that we would not otherwise consider.

5. How could a high school student with knowledge of algebra solve this problem?

   As Dana hints in line 9 of the dialogue, slope plays an important part in this problem. The students in this problem essentially use the coordinate plane and the rise-over-run definition of slope to find the fourth vertex. A high school student with more knowledge of algebra could use the fact that two parallel lines have the same slope and the slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, to calculate the coordinates of the fourth vertex. This type of reasoning would exhibit high school standard G-GPE.B.5, "Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point)." A high school student may also use the coordinates to calculate the distance between the points to establish that the opposite sides of the parallelogram are congruent. Finally, older students may be able to address Sam's question in line 7: "What if the fourth point is not even on a gridline?" High school students may explore parallelograms whose vertices don't have integer coordinate values, and may also explore the question: If a parallelogram has three vertices that have integer coordinate values, is it possible for the fourth vertex not to have integer coordinate values?

6. What did using graph paper as opposed to blank paper allow the students to do? How/why can graph paper be considered a tool? Go back to the dialogue to find evidence of how using graph paper contributed to the students' thinking as they worked to solve the problem.

   In this problem, the students use graph paper not only to plot the three given points on the Cartesian plane, but also as a measuring tool. Once the points are plotted, the students are able to count the number of vertical and horizontal units between two points. We can see this in line 11 when Sam measures the rise and run between points *B* and *C* and then measures out the same rise and run to get from point *A* to the fourth possible vertex. For a middle school student with limited or no formal knowledge of algebra, the use of graph paper would have been vital in allowing them to find the exact coordinate of the fourth vertex.

7. What might be the affordances and the risks in providing Geoboards to the students in the dialogue?

   Using a Geoboard could affect the students' thinking in several ways. First, if students start out with the same two lines ($\overline{AB}$ and $\overline{BC}$), the Geoboard makes it much easier to find the fourth point of the parallelogram. Students would likely not guess an incorrect point, and students might feel like they are "done" with the problem much sooner. Geoboards, however, don't allow for the possibility of Sam's insight in line 7—that the fourth point might not even have integer coordinate values. The Geoboard limits vertices to the gridlines, so students are not led to question their assumption that what "looks" like a parallelogram is the parallelogram they are trying to find. Finally, the students realize at the end of the dialogue that there are more possible parallelograms to explore. This insight may be harder to make using a Geoboard. On graph paper, students can plot the points and then make the separate decision to connect them with lines. On a Geoboard, the line segment made by the rubber band also serves to locate the endpoints, so students may be less likely to identify the decision they made in connecting $\overline{AB}$ and $\overline{BC}$ instead of $\overline{AC}$ and $\overline{BC}$ (or $\overline{AB}$ and $\overline{AC}$).

8. If this dialogue were happening as a classroom discussion, where in the dialogue would you ask students to repeat their thinking for extra emphasis, ask for more clarification, or otherwise draw the class's attention to students engaging in the mathematical practices?

   Responses will vary. To highlight MP 1, we see Sam asking "What do we know about parallelograms" (line 1), Anita proposing "I think there's a way to find the point we want without guessing" (line 8), or Dana wondering if there might be more parallelograms (line 14). To highlight MP 3, we see Sam questioning the logic of judging a parallelogram by how it looks (line 7), Dana asking a clarifying question as Anita describes her method of sliding her hand (line 9), and Anita summarizing and communicating the collective argument (line 12). Lines to emphasize for MP 5 include Dana using a ruler (line 5) and Sam realizing that the fourth point may not even be on a gridline (line 7). Finally, MP 7 is most clearly demonstrated in the thinking of Anita and Dana as they talk about $\overline{BC}$ "sliding over" (lines 8–10) and Sam and Anita showing and explaining how the tilt and length between the points is related to the relative horizontal and vertical distances between them (lines 11–12).

9. Given points (2,1), (6,1) and (2,-2), how many parallelograms can you make? How many rectangles? Explain your reasoning.

   As in the previous problem there are three possible points that could be the fourth vertex of this parallelogram. The possible fourth vertices are (6,4), (6,-2), and (-2,-2). Of these, only one of them is a possible fourth vertex for a rectangle: (6,-2). The shape made by the four vertices must fit the definition of a parallelogram, and it must also have all 90º angles. Out of the three possible parallelogram fourth vertices, only (6,-2) produces a parallelogram with all 90º angles. In this case, showing that the angles are 90º angles is trivial because the edges of the rectangle line up with the vertical and horizontal lines of the grid. In cases where none of the rectangle sides has an undefined slope (i.e., sides aren't vertical), right angles may also be justified by examining the relationship between the slopes of the perpendicular lines. For middle school students, this may be a way to explore the relationship between the tilt of perpendicular lines before they learn a "rule" for the relationship.

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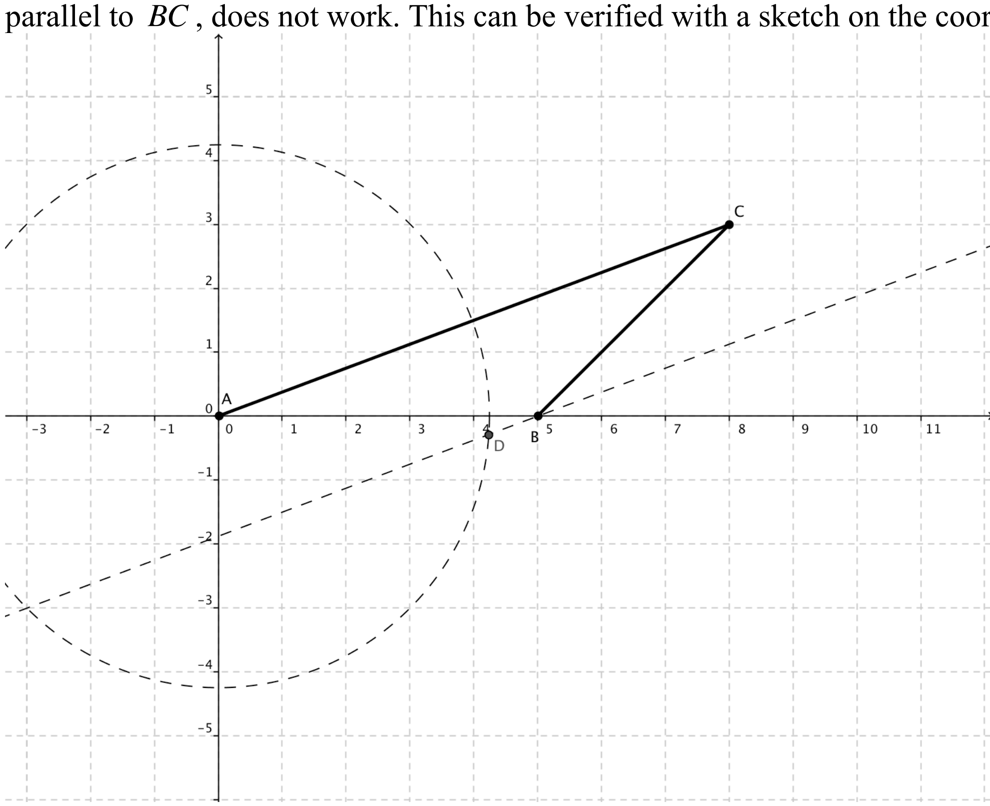
10. If the points $A(0,0)$, $B(5,0)$, and $C(8,3)$ were the vertices of an isosceles trapezoid, how many possible fourth vertices could you have? Justify your answer.

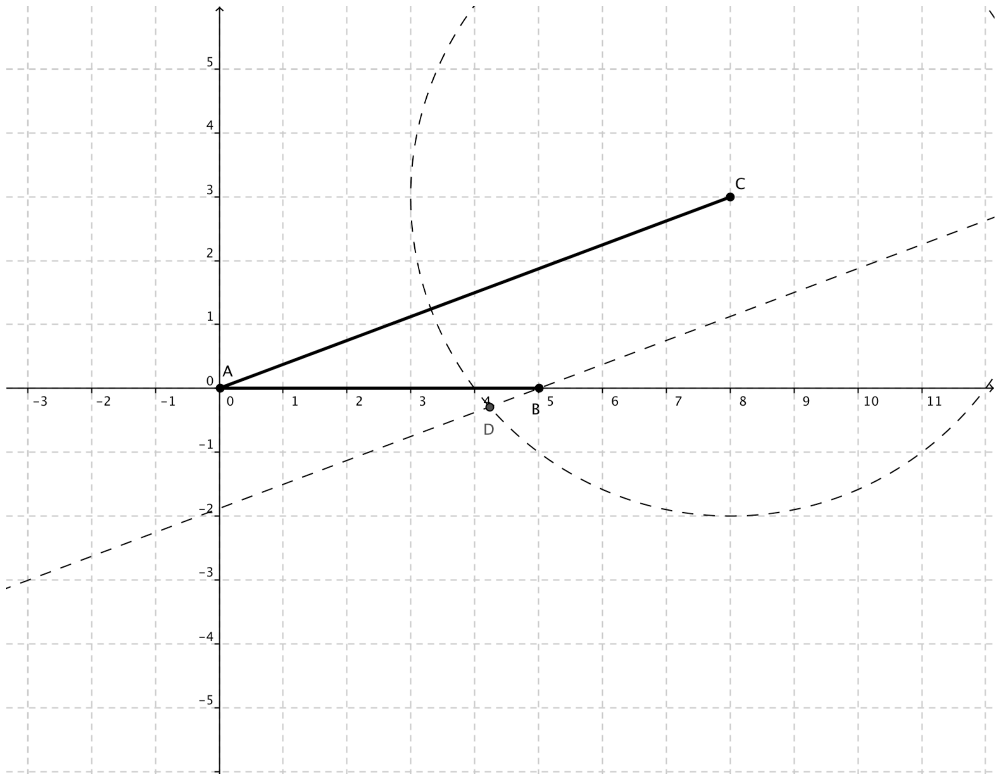
When first looking at this problem, one might be tempted to say that there are three possible locations for the fourth vertex because you can break the problem up into three cases similar to the parallelogram problem. However, unlike the parallelogram problem, not all cases allow for an isosceles trapezoid. Some cases allow for two isosceles trapezoids, some allow for one, and some do not allow for any isosceles trapezoids to be formed.

CASE #1: If you connect the vertices and have $\overline{AB}$ and $\overline{BC}$, that means the fourth vertex, point D, must be located at (-3,3) in order to have $\overline{AD} \cong \overline{BC}$ or at (8,8) to have $\overline{AB} \cong \overline{CD}$.

CASE #2: If you connect the vertices and have $\overline{AC}$ and $\overline{BC}$, the fourth vertex must be at $\left(\frac{309}{73}, -\frac{21}{73}\right)$, which is approximately equal to (4.23, -.29). This point can be found by calculating the point of intersection between a line parallel to $\overline{AC}$ passing through point B (since trapezoids have one set of parallel sides) and a circle around point A with a radius of $\sqrt{18}$ (since isosceles trapezoids have two congruent sides and $\overline{BC}$ has a length of $\sqrt{18}$). Please note that the second orientation, an isosceles trapezoid with a side parallel to $\overline{BC}$, does not work. This can be verified with a sketch on the coordinate plane.



CASE #3: If you connect the vertices to have $\overline{AB}$ and $\overline{AC}$, there is no way to make an isosceles trapezoid (using the definition that trapezoids have exactly one pair of parallel sides). Drawing a line parallel to $\overline{AC}$ and passing through point *B*, and drawing a circle around point *C* with a radius of 5 (the length of $\overline{AB}$), you see that the point of intersection, point *D*, would not form an isosceles trapezoid with the other three given vertices. Please note that the second orientation, an isosceles trapezoid with a side parallel to $\overline{AB}$, also does not work. This can be verified with a sketch on the coordinate plane.



11. If the points *A*(0,0), *B*(5,0), and *C*(8,3) were the vertices of a trapezoid (not necessarily isosceles), how many possible fourth vertices could you have? Justify your answer.

    There are an infinite number of locations where the fourth vertex could be. Once the restriction that the trapezoid must be isosceles is removed, there are an infinite number of places where the fourth vertex may be located along certain lines. For example, if you connect the vertices to have $\overline{AB}$ and $\overline{BC}$, any point on the dotted line parallel to $\overline{AB}$ (in the case that $\overline{AB}$ is a base of the trapezoid) and passing through point *C* whose *x*-coordinate is less than 8 would work. Note, however, that (3,3) produces a parallelogram, and, depending on your definition of a trapezoid, may or may not be counted as a possible fourth vertex. Similar lines of possible points exist in the other cases considered in problem 10.

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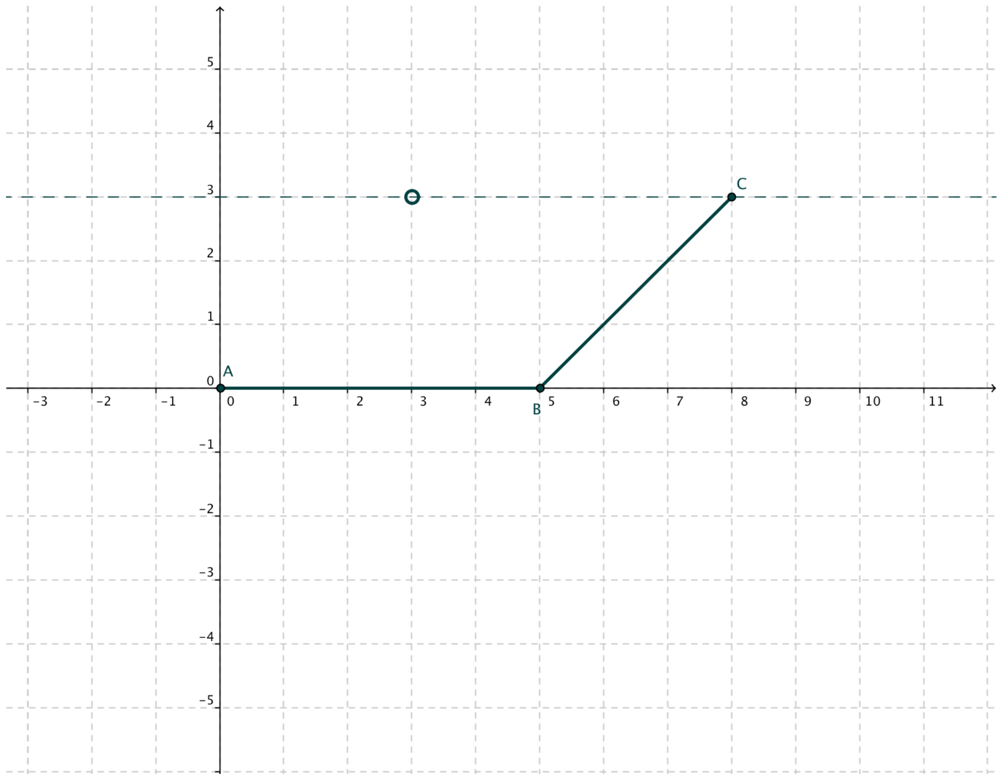


Possible Responses to Student Discussion Questions

1. What does Sam do in line 11 of the dialogue to “measure” the side of the parallelogram? Will this method always work to measure the side of a parallelogram?

   Sam “measures” $\overline{BC}$ by counting the vertical units (“up”) and horizontal units (“over”) between points B and C. By counting, Sam is able to figure out a way to describe how long and slanted $\overline{BC}$ is. This method will always work as long as the vertices are on the gridlines (i.e., have integer coordinate values). The method does work in general, but “counting” is not precise if the points don’t have integer coordinate values. Note that Sam does not find the length of $\overline{BC}$, however. The “up and over” method is enough to produce a line of the same length, but to find the length takes at least one more calculation.

2. In line 11, why does Sam move up 2 units and over 1 unit from point A?

   Sam is trying to make the side opposite to $\overline{BC}$ be the same length and slanted the same way as $\overline{BC}$. In line 8 of the dialogue, Anita visualizes $\overline{BC}$ sliding over to form the opposite side of the parallelogram, and this is what Sam is trying to draw.

3. In line 2, Anita identifies a property of parallelograms that the students use to find the fourth vertex in the dialogue. What is the property? What are other properties of parallelograms that the students could also use?

    A parallelogram is a quadrilateral that can be defined from several perspectives. The students use the property that the opposite sides in a parallelogram are parallel. The students could also use the property that the opposite sides of a parallelogram are congruent (same length), or that the opposite angles of a parallelogram are congruent (same measure).

4. What assumptions do the students make in the beginning of their problem-solving process that Dana discovers in lines 14 and 16? How does making the assumptions affect their problem-solving process?

    The students assume that the parallelogram has $\overline{AB}$ and $\overline{BC}$. By connecting the vertices in this way, they limit themselves to only one place where the fourth vertex can be. The students realize this at the end of the dialogue and must now consider the other parallelograms with the three given vertices: the parallelogram with $\overline{AC}$ and $\overline{BC}$ and the parallelogram with $\overline{AB}$ and $\overline{AC}$. Note that making the assumption is not a bad thing, and is even necessary as a starting place for the problem-solving process.

Possible Responses to Related Mathematics Tasks

1. Find the coordinates of all other possible fourth vertices of the parallelogram described in the problem.

    The possible fourth vertices are (2,4), (0,0), and (8,2).

2. How do you know you have found all the possible fourth vertices? Justify your answer.

    Initially, we are only given three vertices of the parallelogram. Like in the dialogue, we can connect the vertices to start drawing the parallelogram. Considering all the different possible scenarios, we can connect the vertices to get $\overline{AB}$ and $\overline{BC}$ (case #1), $\overline{AC}$ and $\overline{BC}$ (case #2), or $\overline{AB}$ and $\overline{AC}$ (case #3). Another way to justify why there are only three possible points for the fourth vertex is to think about opposite corners of a parallelogram. The fourth vertex of the parallelogram must be opposite some other vertex. Since we have three vertices to choose from, that must mean the fourth vertex may be in one of three places.

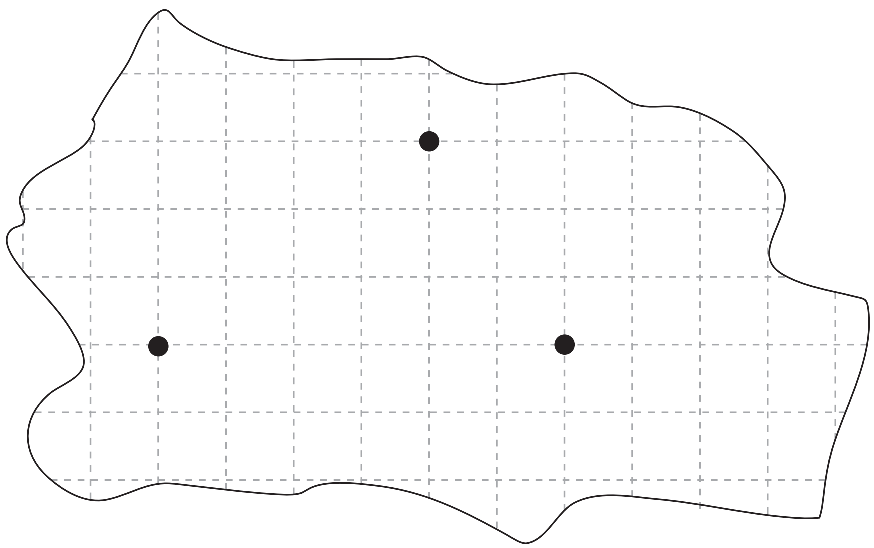
3. Given points (2,1), (6,1), and (2,-2), how many parallelograms can you make? How many rectangles? Justify your answer.

    As in the previous problem there are three possible points that could be the fourth vertex of this parallelogram. The possible fourth vertices are (6,4), (6,-2), and (-2,-2). However, given the three points as the corners of a rectangle, there is only one possible fourth vertex at (6,-2). The reason for this is that not only must the shape made by the four vertices fit the definition of a parallelogram, but it must also have all 90º angles. Out of the three possible parallelogram fourth vertices, only (6,-2) produces a parallelogram with all 90º angles.

4. Name three points that cannot be the vertices of a parallelogram, no matter where the fourth point is. What must be true about these three points?

   Any three collinear points cannot be the vertices of a parallelogram, no matter where the fourth point is. As long as the three points are not collinear, any three points can make up three of the four vertices of a parallelogram.

5. Four keys are required to open a treasure vault. Your ripped map shows the location of three of the keys. You know the four keys are located at the corners of a parallelogram. Where could the fourth key be?



   To solve this problem, students have to use MP 4 (model with mathematics) to label the points and find a way to talk about the location of the fourth key (again, there are three possible locations).