Integer Combinations—Postage Stamps Problem (HS Version)

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Integer Combinations—Postage Stamps Problem (HS Version)* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the amounts of postage that are impossible to make using different pairs of stamp denominations. In the case of five-cent and seven-cent stamps, students notice that all postage after a certain point can be made and explore why. In the case of six-cent and nine-cent stamps, students realize that all postage must contain a factor of three and explain why.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 2: Reason abstractly and quantitatively.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 4: Model with mathematics.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 8–9

Target Content Domain: Seeing Structure in Expressions (Algebra Conceptual Category), Functions

Highlighted Standard(s) for Mathematical Content

A.SSE.A.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.
A.SSE.B.3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
8.F.A.1	Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Math Topic Keywords: equations, algebraic expressions, greatest common factor, relatively prime numbers

This material is based on work supported by the National Science Foundation under Grant No. DRL-1119163. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Part 1: Suppose the post office sold only five-cent stamps and seven-cent stamps. Some amounts of postage can be made with just those two kinds of stamps. For example, 1 five-cent and 2 seven-cent stamps make 19 cents in postage, and 2 five-cent stamps makes 10 cents in postage. Which amounts of postage is it impossible to make using only five-cent and seven-cent stamps?

Part 2: Suppose the post office only sold six-cent and nine-cent stamps. Which amounts of postage is it impossible to make?

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students in this dialogue have been studying the properties of arithmetic and have worked extensively with addition tables. They are now exploring what numbers can and cannot be produced by adding only two types of numbers.

(1) Chris: Well, if we want to figure out what postage can't be made, maybe we should make a list of all the postage that *can* be made first.

(2) Lee: That's not a bad idea. And we know that all our postage is made using only five-cent and seven-cent stamps so… wouldn't all the possible postage have to equal $5x+7y$?

(3) Matei: That's right. That expression will give us the possible postage values, but we should probably find the actual numbers that can be made. Let's plug in different values for x and y to see what postage values we can make. Actually… why don't we use a table to show the different combinations we can make? We can have the number of five-cent stamps going across and the number of seven-cent stamps going down.

(4) Chris: That sounds like a good idea!

[Students take a few minutes and create the following.]

		5-Cent Stamps							
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
7-Cent Stamps	**0**	0	5	10	15	20	25	30	35
	1	7	12	17	22	27	32	37	42
	2	14	19	24	29	34	39	44	49
	3	21	26	31	36	41	46	51	56
	4	28	33	38	43	48	53	58	63
	5	35	40	45	50	55	60	65	70
	6	42	47	52	57	62	67	72	77
	7	49	54	59	64	69	74	79	84

(5) Lee: Great! Now let's see what numbers we have and which ones we don't.

(6) Chris: Well, we've got 5… I don't see 6 but we have 7 here… no 8 or 9… we've got 10 but I don't see 11.

[A few minutes pass in which students are looking to see which values are in the table and which are not.]

No to 23. I see 24 here, and 25, 26 and 27, 28, 29, 30, 31… Hmm, it doesn't seem like we are skipping over any numbers now. I guess we can make any postage greater than 23.

(7) Lee: That's strange! Why's that?

(8) Chris: Well…. I'm not sure, but maybe it has something to do with the way I'm moving on the table? To go from one number to the next, I keep going right and up or down and left.

(9) Lee: You might be right! Check this out: If you start at 31, go to the right three squares then up two squares, you get to 32. *[draws green arrows to represent the described path—see figure below]* It even works with other numbers. If you start with 38 and go right three and up two, you've got 39.

		5-Cent Stamps							
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
7-Cent Stamps	**0**	0	5	10	15	20	25	30	35
	1	7	12	17	22	27	32	37	42
	2	14	19	24	29	34	39	44	49
	3	21	26	31	36	41	46	51	56
	4	28	33	38	43	48	53	58	63
	5	35	40	45	50	55	60	65	70
	6	42	47	52	57	62	67	72	77
	7	49	54	59	64	69	74	79	84

(10) Matei: I think that has to do with what it means to move on the table. If we moved to the right 3, that means we are adding 3 five-cent stamps. And if we are moving up 2, we are taking away 2 seven-cent stamps. That's like saying:

$31+3\cdot 5-2\cdot 7=$

$31+(15-14)=$

$31+1=32$

(11) Chris: But that doesn't work for 58. You'd be going off the table moving to the right and up.

(12) Lee: Well, we can just extend the table. I mean, we can ask for more than 7 five-cent or seven-cent stamps.

(13) Chris: You're right, that makes sense. Oh! But wait, we've got a 59 right here, too! There it is—down three squares from 58, then to the left four squares.

[draws red arrows to represent the path]

7-Cent Stamps \ 5-Cent Stamps	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	5	10	15	20	25	30	35
1	7	12	17	22	27	32	37	42
2	14	19	24	29	34	39	44	49
3	21	26	31	36	41	46	51	56
4	28	33	38	43	48	53	58 ↓	63
5	35	40	45	50	55	60	65 ↓	70
6	42	47	52	57	62	67	72 ↓	77
7	49	54	59	← 64	← 69	← 74	← 79	84

(14) Matei: Right, because we're doing:

$$58 + 3 \cdot 7 - 4 \cdot 5 =$$
$$58 + (21 - 20) =$$
$$58 + 1 = 59$$

(15) Chris: So what does all this tell us?

(16) Lee: Well, it looks like we found two ways to add 1 to a postage amount. So I guess we can build all numbers after 23.

(17) Matei: I'm convinced it works for all numbers after 23.

(18) Chris: Me too. What about the next part, six-cent and nine-cent stamps?

(19) Lee: 6 and 9 seem really different to me from 5 and 7. 5 and 7 don't have any common factors except for 1.

(20) Matei: Right. Their greatest common factor is 3, not 1. 6 is 2×3; 9 is 3×3. That means, any postage amount you can make by combining six- and nine-cent stamps will be some number m times 6 plus some number n times 9. *[writes* $6m + 9n$ *]* This is the same as $3(2m + 3n)$.

(21) Chris: How does that help us?

(22) Matei: It says that if we made a 6 by 9 table like we did the 5 by 7 table, all the numbers on it would be multiples of 3. So, the final postage amount will always contain a factor of 3.

(23) Chris: Will all multiples of 3 be on the table?

(24) Lee: You mean, except for 3?

(25) Chris: Yeah, except for 3.

(26) Lee: I…think…so.

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. What mathematics in this dialogue is most likely to be confusing for students?

3. Suppose the post office only sold two-cent and three-cent stamps. After which amount will all postage values be possible? What if they sold three-cent and five-cent stamps? Or four-cent and nine-cent stamps?

4. What conjectures do you have about what characterizes two postage denominations, M and N, for which all postage values after a certain point can be made? What conjectures do you have about what characterizes two postage denominations, M and N, for which there will always be postage values that can't be made?

5. Chris asks a generalizing question at the end. "Will all multiples of 3 be on the [6 by 9] table?" Will they *all* be on the table (except for 3)?

6. If you were the teacher listening to this exchange, what might you ask or say at the end of the exchange to help Lee become more certain.

7. If only two-cent and three-cent stamps were sold, what movements would take you to the next consecutive number, given a table of postage produced from those denominations similar to the one in the dialogue?

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Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
Make sense of problems and persevere in solving them.	The students look for an entry point into the problem by organizing their data in a (+5) by (+7) table so that they can determine what values of postage they can and cannot make using only five- and seven-cent stamps. When a pattern emerges and the students notice that all postage values after 23 are possible (line 6), they ask why and persevere in coming up with an adequate explanation based on the movement along the table they've constructed.
Reason abstractly and quantitatively.	Matei abstracts from the context of postage stamps to develop numerical procedures that show how to proceed on the table from one postage value to the next greater number (lines 10, 14). Matei's ability to seamlessly move from the tabular/geometric model to numeric expressions demonstrates that the student can decontextualize or abstract from the scenario in order to explain why the numbers are increasing by 1 as a result of specific movements on the table.
Construct viable arguments and critique the reasoning of others.	Chris tries to find a counterexample to show that moving three to the right and two up on the table from 58 is not possible (line 11). This, however, ends up being incorrect and Chris' argument is critiqued by Lee (line 12). Another example of viable argument occurs in Matei's two numeric explanations of why the arrow paths yield a +1 effect (lines 10, 14). Lastly, students make another viable argument in showing that in the six-cent by nine-cent example, all entries must be multiples of 3 and so cannot contain all numbers after any point.
Model with mathematics.	Students are able to model the possible postage amounts using a table. Building this model shows the students' ability to identify the constraints of the problem (they can use only five- and seven-cent stamps) and understand the scenario (using different combinations of the two types of stamps in making postage). Students use the model to draw conclusions (line 6) and provide an explanation for the postage amounts that can be made (lines 9–17).

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Look for and express regularity in repeated reasoning.	Chris notices that all numbers greater than 23 can be found on the table and conjectures that this might have something to do with the regularity of movement on the table (right & up and down & left) that were used to go from one number to its consecutive (lines 6, 8). Lee then sees that the green-arrow and red-arrow paths can be repeated over and over again to guarantee that all postage values after 23 cents will be on the table (lines 9, 12, 16). In lines 10 and 14, Matei expresses the regular movements Lee found on the table using numerical expressions that are based on the context of the problem and the meaning behind the table. Another example of the students looking for regularity can be seen when Chris asks the generalizing question at the end, “Will all multiples of 3 be on the table?” (line 23).

Commentary on the Mathematics

This problem is most evidently about integer equations of the form $rM + sN = P$, where r and s are integers ≥ 0, and M and N are whole numbers. Also elicited by the problem, as Matei demonstrates a couple of times, are algebraic expressions and equivalence, as can be seen in lines 10 and 14). Matei demonstrates numerically why the geometric paths that Lee and Chris use show a way to go from one number to that number plus 1. Characterizing integers M and N such that $rM + sN = 1$ relates to the Euclidean algorithm, derived from Euclid and one of the most important calculations in arithmetic. For any two integers, the algorithm computes the greatest common factor (GCF). If the two numbers, r and s, are relatively prime, then the GCF is 1, and that fact can be used to show that there are integers M and N such that $rM + sN = 1$. This is the relationship Lee and Chris capitalize on in moving between consecutive numbers on the table and Matei capitalizes on in developing numerical expressions that have a +1 effect. The postage stamps problem is both rich and deep: younger students can explore number combinations and exercise their algebraic thinking, and students all the way into graduate school can explore generalized aspects of the problem.

<u>Evidence of the Content Standards</u>

In lines 10 and 14, Matei writes numerical expressions for the movements between two consecutive numbers observed in the table. By simplifying and rewriting the expressions with parenthesis (A.SSE.A.2), Matei is able to highlight why those movements produce a +1 effect. In line 20 and 22, Matei takes the expression $6m + 9n$ and rewrites it to the equivalent expression $3(2m + 3n)$ to show why all postage made from six- and nine-cent stamps will be a multiple of 3 (A.SSE.B.3). Students in this dialogue are also implicitly making use of the fact that $5x + 7y$ is a function and that varying the values of x and y (the number of five- and seven-cent stamps respectively) will yield all the possible postage amounts that can be made using those stamps. They understand that for each pair of inputs, only one output will be given (8.F.A.1).

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. A. Find a combination of five-cent stamps and seven-cent stamps that equals 61 cents in postage. Find a combination that equals 89 cents.
   B. Find a combination of four-cent stamps and nine-cent stamps that equals 61 cents. Find a combination that equals 89 cents.

2. In line 22 of the student dialogue, Matei claims that every value on a 6 by 9 table would be a multiple of 3. Is this true? Why?

3. Assuming m and n can be any integer ≥ 0, what values can $2m+3n$ have?

4. Based on your answer to question 3, what values can the expression $3(2m+3n)$, which Matei found in line 20, have? What does this tell you about the postage that can be made using only six-cent and nine-cent stamps?

5. A. In the dialogue, students find movements on the table (described as changes in the number of stamps bought of each denomination) that get them to the next consecutive postage value. They then explain the effect of this movement by showing it corresponds to a numerical expression equal to 1. What is one such movement and its corresponding expression?
   B. If the post office only sold two-cent and three-cent stamps, what movement would cause an increase of 1 in postage? What is its corresponding expression?
   C. If the post office only sold three-cent and seven-cent stamps, what movement would cause an increase of 1 in postage? What is its corresponding expression?
   D. If the post office only sold five-cent and thirteen-cent stamps, what movement would cause an increase of 1 in postage? What is its corresponding expression?

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Related Mathematical Tasks

1. If the post office has only two-cent stamps and three-cent stamps, what amounts of postage cannot be made?

2. If the post office has only three-cent stamps and five-cent stamps, what amounts of postage cannot be made?

3. If the post office has only four-cent stamps and nine-cent stamps, what amounts of postage cannot be made?

4. If the post office has only four-cent stamps and six-cent stamps, what amounts of postage cannot be made?

5. Suppose the post office sells only m-cent stamps and n-cent stamps. Suppose also that, above some amount of postage, all amounts of postage can be made. What can you say about m and n?

6. What was the largest impossible amount of postage in each of the questions 1–3? For m-cent stamps and n-cent stamps in which all postage after a certain point can be made, what is the largest impossible amount of postage in terms of m and n?

7. How would the original task from the dialogue change if you could buy a negative number of stamps? (This can also be thought of as buying stamps worth –5 or –7 cents.) Which amounts of postage are impossible to make using only five-cent and seven-cent stamps? Which amounts of postage are impossible to make using only six-cent and nine-cent stamps?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. What mathematics in this dialogue is most likely to be confusing for students?

   One area of possible confusion is Matei's use of expressions to describe the movement on the table from one number to the next consecutive number. These expressions require students to think about the relationship between moving across cells in the grid and the value of postage. Students then must translate that relationship into a series of operations and numbers in order to write an expression. Another area in which students may struggle is in understanding how the characters in the dialogue come up with the expression $6m+9n$, as well as the equivalent expression $3(2m+3n)$, and how they interpret the possible values that such an expression can produce.

3. Suppose the post office only sold two-cent and three-cent stamps. After which amount will all postage values be possible? What if they sold three-cent and five-cent stamps? Or four-cent and nine-cent stamps?

   All postage values greater than 1 can be made using two-cent and three-cent stamp combinations. All postage values greater than 7 can be made using three-cent and five-cent combinations. All postage values greater than 23 can be made using four-cent and nine-cent stamps.

4. What conjectures do you have about what characterizes two postage denominations, M and N, for which all postage values after a certain point can be made? What conjectures do you have about what characterizes two postage denominations, M and N, for which there will always be postage values that can't be made?

   If M and N are relatively prime (meaning that they have no common factor greater than 1) there will be a number beyond which M and N generate all postage values. If however, M and N have a common factor other than 1, then all possible postage values will also need to have that common factor, and this will lead to an infinite number of impossible postage amounts.

5. Chris asks a generalizing question at the end. "Will all multiples of 3 be on the [6 by 9] table?" Will they *all* be on the table (except for 3)?

    Yes. Any number N on the table will be of the form $N = 3(2m + 3n)$ for some non-negative integers n and m. Non-negative integer combinations of 2 and 3 generate all whole numbers greater than 1 (a fact that may be easier to see if you think of 3 as 2+1), which means the expression $2m + 3n$ can yield all integers 2 and greater and, therefore, N can yield all multiples of 3 that are 6 or greater.

6. If you were the teacher listening to this exchange, what might you ask or say at the end of the exchange to help Lee become more certain.

    A teacher might suggest that since the students got so much from their 5 by 7 table, they try a 6 by 9 table. That might give them some empirical data to conjecture with. From there, it would be a matter of teacher judgment to decide whether to prompt a viable argument to show the conjecture is true.

    Alternatively, students could be encouraged to more closely analyze the meaning of the expression $3(2m + 3n)$. While it might be simpler to have them understand why the value of the expression *must be a multiple* of 3, it is also possible to convince them that all multiples of 3 *will be generated* by the expression as well (except for 3). To support students, have them try out small values for m and n and see what the expression $2m + 3n$ yields. Once they are convinced that all values 2 and greater will be produced, they will understand why all multiples of 3 that are 6 or greater will be generated by $3(2m + 3n)$; see Student Discussion Questions for more support.

7. If only two-cent and three-cent stamps were sold, what movements would take you to the next consecutive number, given a table of postage produced from those denominations similar to the one in the dialogue?

    To get a movement that will take you to the next consecutive number in the table, the equation $2x + 3y = 1$ must be satisfied where x and y are the number of two-cent and three-cent stamps, respectively. One such movement could be adding 3 three-cent stamps and subtracting 4 two-cent stamps ($2 \cdot (-4) + 3 \cdot 3 = 1$). Another possible movement would be subtracting 7 two-cent stamps and adding 5 three-cent stamps ($2 \cdot (-7) + 3 \cdot 5 = 1$). Note that this second movement is similar to the expression $3 \cdot 5 - 2 \cdot 7$ used by Matei in line 10. In other words, $3 \cdot 5 - 2 \cdot 7 = 1$ can be interpreted as a movement of $(3, -2)$ on a table of five- and seven-cent stamps or a movement of $(5, -7)$ on a table of three- and two-cent stamps.

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Possible Responses to Student Discussion Questions

1. A. Find a combination of five-cent stamps and seven-cent stamps that equals 61 cents in postage. Find a combination that equals 89 cents.
   B. Find a combination of four-cent stamps and nine-cent stamps that equals 61 cents. Find a combination that equals 89 cents.

   A. 61 cents of postage can be made using 8 five-cent stamps and 3 seven-cent stamps ($8 \cdot 5 + 3 \cdot 7 = 61$). 89 cents of postage can be made using 15 five-cent stamps and 2 seven-cent stamps ($15 \cdot 5 + 2 \cdot 7 = 89$).
   B. 61 cents of postage can be made using 13 four-cent stamps and 1 nine-cent stamp ($13 \cdot 4 + 1 \cdot 9 = 61$). 89 cents of postage can be made using 20 four-cent stamps and 1 nine-cent stamp ($20 \cdot 4 + 1 \cdot 9 = 89$).

2. In line 22 of the student dialogue, Matei claims that every value on a 6 by 9 table would be a multiple of 3. Is this true? Why?

   Yes, every value on the table would be a multiple of 3 because both 6 and 9 are multiples of 3 so every number produced by combinations of 6 and 9 would also have to be a multiple of 3. Alternatively, looking at the expression $3(2m + 3n)$, which describes the values on the 6 by 9 table, you can see that every number will have 3 as a factor and, therefore, be a multiple of 3.

3. Assuming m and n can be any integer ≥ 0, what values can $2m + 3n$ have?

   By substituting in different values for m and n, you can see that $2m + 3n$ can be 0 or any integer ≥ 2.

4. Based on your answer to question 3, what values can the expression $3(2m + 3n)$, which Matei found in line 20, have? What does this tell you about the postage that can be made using only six-cent and nine-cent stamps?

   Since $2m + 3n$ can be 0 or any integer ≥ 2, that must mean $3(2m + 3n)$ must be 0 or any multiple of 3 that is 6 or greater.

5. A. In the dialogue, students find movements on the table (described as changes in the number of stamps bought of each denomination) that get them to the next consecutive postage value. They then explain the effect of this movement by showing it corresponds to a numerical expression equal to 1. What is one such movement and its corresponding expression?
   B. If the post office only sold two-cent and three-cent stamps, what movement would cause an increase of 1 in postage? What is its corresponding expression?
   C. If the post office only sold three-cent and seven-cent stamps, what movement would cause an increase of 1 in postage? What is its corresponding expression?
   D. If the post office only sold five-cent and thirteen-cent stamps, what movement would

cause an increase of 1 in postage? What is its corresponding expression?

A. In the dialogue students either added 3 five-cent stamps and took away 2 seven-cent stamps (corresponding to $3 \cdot 5 - 2 \cdot 7$) or added 3 seven-cent stamps and took away 4 five-cent stamps (corresponding to $3 \cdot 7 - 4 \cdot 5$).
B. In the case of two- and three-cent stamps, adding 1 three-cent stamp and taking away 1 two-cent stamp would cause an increase of 1 cent in postage. This corresponds to the expression $1 \cdot 3 - 1 \cdot 2$. Other movements may exist.
C. In the case of three- and seven-cent stamps, adding 1 seven-cent stamp and taking away 2 three-cent stamps would cause an increase of 1 cent in postage. This corresponds to the expression $1 \cdot 7 - 2 \cdot 3$. Other movements may exist.
D. In the case of five- and thirteen-cent stamps, adding 2 thirteen-cent stamps and taking sway 5 five-cent stamps would cause an increase of 1 cent in postage. This corresponds to the expression $2 \cdot 13 - 5 \cdot 5$. Other movements may exist.

Possible Responses to Related Mathematical Tasks

1. If the post office has only two-cent stamps and three-cent stamps, what amounts of postage cannot be made?

Total Postage	How?
1	Impossible
≥ 2	Possible

2. If the post office has only three-cent stamps and five-cent stamps, what amounts of postage cannot be made?

Total Postage	How?
1	Impossible
2	Impossible
3	Possible; 3
4	Impossible
5	Possible; 5
6	Possible; $3+3$
7	Impossible
≥ 8	Possible

3. If the post office has only four-cent stamps and nine-cent stamps, what amounts of postage cannot be made?

Total Postage	How?
1	Impossible
2	Impossible
3	Impossible
4	Possible; 4

5	Impossible
6	Impossible
7	Impossible
8	Possible; $4+4$
9	Possible; 9
10	Impossible
11	Impossible
12	Possible; $4+4+4$
13	Possible; $4+9$
14	Impossible
15	Impossible
16	Possible; $4+4+4+4$
17	Possible; $4+4+9$
18	Possible; $9+9$
19	Impossible
20	Possible; $4+4+4+4+4$
21	Possible; $4+4+4+9$
22	Possible; $4+9+9$
23	Impossible
≥ 24	Possible

4. If the post office has only four-cent stamps and six-cent stamps, what amounts of postage cannot be made?

Total Postage	**How?**
1	Impossible
2	Impossible
3	Impossible
4	Possible; 4
5	Impossible
6	Possible; 6
7	Impossible
8	Possible; $4+4$
9	Impossible
10	Possible; $4+6$
11	Impossible
12	Possible; $6+6$
13	Impossible
14	Possible; $4+4+6$

All multiples of 2 that are greater than or equal to 4 can be made. All other postage values cannot be made.

5. Suppose the post office sells only m-cent stamps and n-cent stamps. Suppose also that, above some amount of postage, all amounts of postage can be made. What can you say about m and n?

   m and *n* do not have a common factor other than 1. This is called "relatively prime."

6. What was the largest impossible amount of postage in each of the questions 1–3? For m-cent stamps and n-cent stamps in which all postage after a certain point can be made, what is the largest impossible amount of postage in terms of m and n?

First Stamp Value	**Second Stamp Value**	**Largest Impossible Amount of Postage**
2	3	1
3	5	7
4	9	23
m	n	$m \cdot n - m - n$

7. How would the original task from the dialogue change if you could buy a negative number of stamps? (This can also be thought of as buying stamps worth –5 or –7 cents.) Which amounts of postage are impossible to make using only five-cent and seven-cent stamps? Which amounts of postage are impossible to make using only six-cent and nine-cent stamps?

   If you can buy a negative number of stamps, that means you are able to extend the table the students started to the other three quadrants (see below) and produce negative amounts of postage. Extending the table allows you to use the "to the right three and up two" and the "down three and to the left four" movements that produce a +1 effect on the postage value. Reversing this movement, in turn, produces a –1 effect on the postage value. This means starting at 0 you can make all consecutive integers and that no integer postage value is impossible.

5-Cent Stamps

7-Cent Stamps

–36	–31	–26	**–3**	–21	–16	–11	–6
–29	–24	–19	**–2**	–14	–9	–4	–1
–23	–17	–12	**–1**	–7	–2	3	8
–3	**–2**	**–1**		**0**	**1**	**2**	**3**
–15	–10	–5	**0**	0	5	10	15
–8	–3	2	**1**	7	12	17	22
–1	4	9	**2**	14	19	24	29
6	11	16	**3**	21	26	31	36

Note: The numbers in the gray boxes indicate the number of five- and seven-cent stamps *and the numbers in the white boxes indicate the total postage formed using a given combination of stamps.*

In the case of six-cent and nine-cent stamps, the postage that can be made would still be given by the expression $3(2m+3n)$. However, in this case m and n are not limited to whole numbers but rather all integers. This would allow you to make any postage that is a positive or negative multiple of 3 (or 0 postage). Note that in this case, you can even make postage of 3 if $n=1$ and $m=-1$; in the dialogue, this isn't possible since you can't buy a negative number of stamps.