# Isosceles Triangles on a Geoboard

**About Illustrations:** Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

**About the *Isosceles Triangles on a Geoboard* Illustration:** This Illustration's student dialogue shows the conversation among three students who are looking for how many isosceles triangles can be constructed on a geoboard given only one side. After exploring several cases where the known side is one of the congruent legs, students then use the known side as the base and make an interesting observation.

**Highlighted Standard(s) for Mathematical Practice (MP)**
MP 1: Make sense of problems and persevere in solving them.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

**Target Grade Level:** Grades 6–7

**Target Content Domain:** Geometry

**Highlighted Standard(s) for Mathematical Content**
7.G.A.2 Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

**Math Topic Keywords:** isosceles triangles, geoboard



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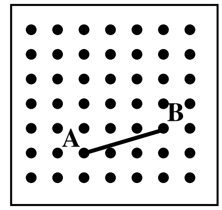
## Mathematics Task

**Suggested Use**
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.




Make an isosceles triangle ( $\triangle ABC$ ) on your geoboard using $\overline{AB}$ as one of the sides. How many different isosceles triangles can you make on your geoboard with $\overline{AB}$ as one of the sides?

Task Source: Adapted slightly from Chicago Lesson Study Group. See http://www.lessonstudygroup.net/lg/lesson_plan_viewer.php?lid=68&n=0

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## Student Dialogue

**Suggested Use**
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

*Students have worked with geoboards on many levels and have had experience with the idea of lattice points. They are in the midst of a unit on exploring triangles in different configurations (equilateral, scalene, isosceles, right, etc.).*

(1) Sam: I see one… just go 4 up from $A$ and make that $C$. *[see Figure 1]*

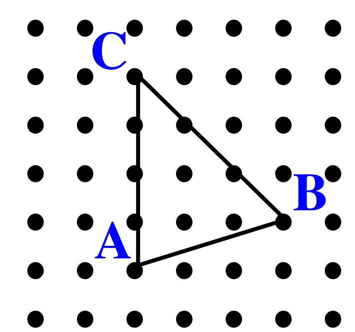

Figure 1

(2) Anita: Is that really right? The distance from $C$ to $B$ is only three steps, and from $C$ to $A$, it's 4.

(3) Dana: Yeah, but if you go straight up, the steps are shorter than if you go diagonally, so the distance from $A$ to $C$ really *might* be the same as the distance from $B$ to $C$.

(4) Anita: But we don't know. So if we want to be sure we have an isosceles triangle, we need to make another side on the geoboard that is the same length as $AB$.

(5) Sam: What's the length of $AB$?

(6) Dana: I don't know, but I know that I can make it by going over 3 and up 1 on the geoboard.

(7) Anita: But that only works to get to $B$. But, from point $A$ we could go up 3 and over 1 and call that point $C$. That would make side $AC$ = side $AB$, and triangle $ABC$ would be isosceles. *[see Figure 2]*

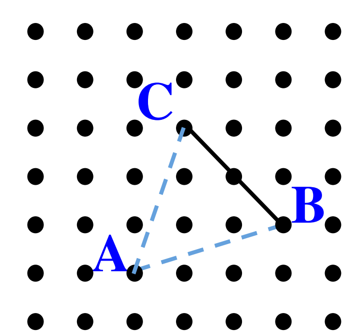

Figure 2

(8) Sam: Oh, wait! Couldn't you make another one by going down three from $A$ and over to the right 1 space?

(9) Dana: Nice! But it goes off the geoboard. Does it count?

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(10) Sam: Why not? But anyway, we can go over 3 and down 1 to stay on the board. *[see Figure 3]* … I was also thinking that we could use that up-3-and-over-1 strategy again but instead go up 3 from $A$ and over 1 to the left to make side $AC$. *[left for the reader to visualize or draw]*

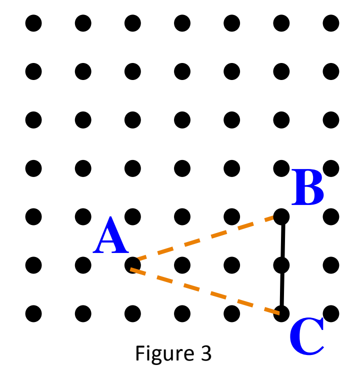

Figure 3

(11) Anita: How many do we have now? And I think there are still others. What if the two sides that are equal are not $AB$ and $AC$ but $AB$ and $BC$?

(12) Sam: Yeah, that's a good idea… we can go up 3 and left 1 from $B$ instead of $A$! *[see Figure 4]*

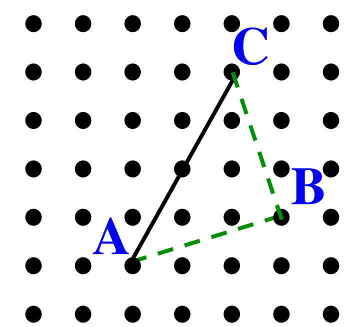

Figure 4

(13) Dana: AND… up 3 and over 1, too. Left AND right. *[left for the reader to visualize or draw]*

(14) Anita: Good, we've got something like 6 or 7 now, but I still think there are more. What if $AC$ and $BC$ are the two sides that are equal?

(15) Dana: Okay, let's try this… if we go up 2 and right 1 from $A$, we end up in the same place as up 1 and over 2 from $B$, so those are the same length. *[see Figure 5]*

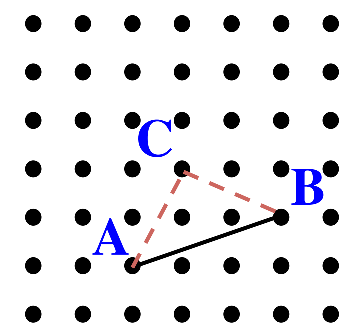

Figure 5

(16) Anita: Yeah, and we can go down 1 and right 2 from $A$, too, which would be down 2 and left 1 from $B$… *[left for the reader to visualize or draw]*

(17) Sam: Well, if you stretch out point $C$, couldn't you make a lot of triangles where the length of $AC$ and $BC$ are equal?

## Teacher Reflection Questions

**Suggested Use**
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. Clearly, Sam (line 17) doesn't mean to "stretch a point." What *is* a clear way of saying what Sam means?

3. If Sam is right, and we can "stretch" point $C$, are there any points on the board that could be point $C$? If so, how would you prove that the point you found *is* equidistant from $A$ and $B$?

4. How are the students using structure (MP 7) in this dialogue to support their conjectures?

5. Why does the up-and-over strategy work as a proof?

6. If we remove the constraint of using only the pegs on the geoboard, where are all the points that would create an isosceles triangle with $\overline{AB}$?

## Mathematical Overview

| **Suggested Use**<br>The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes. |
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***Commentary on the Student Thinking***

| Mathematical Practice | Evidence |
|---|---|
| Make sense of problems and persevere in solving them. | The students in this dialogue are making sense of the problem in several ways. They begin by considering the meaning of isosceles and, in a few cases, consider the constraints of the geoboard ("but it [the point] goes off the geoboard," line 9). They also consider ALL pairs of sides ($\overline{AC}$ and $\overline{BC}$, $\overline{AB}$ and $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$) to try to find all the possible triangles. Sam (line 17) considers the constraint of points $A$ and $B$ being fixed, and realizes that point $C$ is not constrained that way so proposes that many triangles might be possible. |
| Use appropriate tools strategically. | The geoboard is the most prominent tool in use in this problem. It helps the students visualize the triangles and plays into their making sense of the problem and seeing the constraints as well as the possibilities for isosceles triangles. |
| Look for and make use of structure. | Students are using the structure of what an isosceles triangle is, considering that two sides must be of equal length. This use of structure helps them consider all possible pairings of sides (see MP 1 notes, above). Students are also making use of structure through exploration of similar triangles. When going "over 3 and up 1," Dana (line 6) describes the legs of a right triangle whose hypotenuse is the same length as $\overline{AB}$. Without stating it, Dana uses the fact that all right triangles with legs of 3 and 1 have the same length hypotenuse. More likely, students at this age, who have not studied triangle congruence, will think of all 3×1 rectangles, regardless of orientation, as "the same," and will regard these lengths as diagonals of "the same rectangle" and, for that reason (and not triangle congruence reasons), equal lengths. The students repeat that reasoning with "up 2 over 1" (lines 15–16). They broaden this use of structure when Sam (line 17) "stretches out" point $C$, recognizing that there are a lot of places where point $C$ could be that would be equidistant from points $A$ and $B$. |

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***Commentary on the Mathematics***

The students in the dialogue explicitly make use of the definition of isosceles triangles, and some informal ideas about distance on a lattice. As mentioned in the discussion of structure (MP 7), students are intuitively making use of the fact that all rectangles with sides of 3 and 1 are "the same," so their diagonals are "the same," without stating that fact formally. Also intuitively, they use the fact that the hypotenuse of a right triangle is longer than the legs when Dana (line 3) says that going straight up is shorter than going diagonally.

There is more mathematics involved than just isosceles triangles. The problem is confined to the lattice points on the coordinate grid (and, perhaps, just those lattice points that are within the borders of the geoboard). Sam (line 17) begins to explore past this constraint when proposing that point $C$ could be "stretched." The students find two points on the geoboard that are equidistant from points $A$ and $B$ (so that $\overline{AC}$ and $\overline{BC}$ are the same length). If these two points are connected and that connecting line continues infinitely in each direction, *any* point on that line, regardless of whether it is a lattice point or not, would create an isosceles triangle with points $A$ and $B$ (except for the midpoint of $\overline{AB}$). But the problem asks how many "on your geoboard," and students restrict themselves accordingly. It turns out that Sam's line does intersect one other point on the geoboard (see Figure 6). Students (correctly) feel that this point creates an isosceles triangle, but since it is directly above point $A$, we can't rely on the up-and-over method that the students have been using to compare its distance from $A$ and $B$, so we have to turn to another method for proof. The Pythagorean Theorem does this for us. $\overline{BC}$ is the hypotenuse of a right triangle with legs of 3 and 4, which makes $\overline{BC}$ length 5, which is the length of $\overline{AC}$ as well. Because we may use only the lattice points *on the board*, this reasoning of placing point $C$ on the line allows for only three positions. However, if this constraint of using lattice points is removed—Sam's image of stretching seems to suggest a kind of continuous movement—we now have an infinite set of possible positions for point $C$. Good luck getting a rubber band to stick to most of them, though.

<u>Evidence of the Content Standards</u>

The content standard associated to this task is about identifying and constructing geometric shapes from given conditions, particularly focusing on triangles with given side lengths and angle measures, and noticing when these conditions could describe one, more than one, or no triangles. In the task, students are given the conditions of isosceles and the length of one side. The students come to the conclusion that the given side could be one of the two equal sides or the third side. Though it's not specifically stated in the dialogue, clearly they are noticing that these conditions give rise to many triangles.

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## Student Materials

**Suggested Use**
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

***Student Discussion Questions***

1. Discuss why the up-and-over technique works to prove that the lengths are the same.

2. Have the students in the dialogue found all of the isosceles triangles possible?

3. What does Sam mean in line 17 by "stretch out point $C$"? If you did what Sam means, are there any points on the geoboard that you could reach exactly?

***Related Mathematics Tasks***

1. How would the results of the original problem change if you were allowed to put more than one geoboard together?

2. If you are free to move point $A$ (leaving point $B$ where it is), where would you move $A$ to maximize the number of possible locations for point $C$ on the geoboard? What if you could move point $B$ (leave point $A$)?

3. Let's say that you can move both points, but had to leave them exactly the same distance from each other as they are now. In what position(s) can you place them to *maximize* the number of isosceles triangles that can be made?

4. In what position(s) can you place $\overline{AB}$ to *minimize* the number of isosceles triangles that can be made?

5. Imagine the exact same problem posed on another geoboard of the exact same area, but not the exact same shape. How would that affect the number of possible isosceles triangles?

6. How does the problem change if we want an equilateral triangle instead of isosceles? What if we want to guarantee a scalene triangle?

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## Answer Key

**Suggested Use**
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

***Possible Responses to Teacher Reflection Questions***

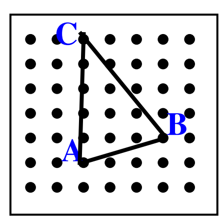
1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. Clearly, Sam (line 17) doesn't mean to "stretch a point." What *is* a clear way of saying what Sam means?

   Students who say things like this are not confused about the concept; they know that points don't stretch and are referring to the distances, which can. Students often draw language from analogy, and the geoboard context makes it natural to picture the edges of this triangle as rubber bands. "If you stretch out point $C$" becomes Sam's way of saying "if you move point $C$ farther away from $\overline{AB}$ in exactly the same direction." At this level, that's good enough. Even greater precision would require specifying what is meant by "same direction." Distance to a line (segment) is measured perpendicular to that line. Moreover, to maintain equal distances from points $A$ and $B$, point $C$ must move along the perpendicular bisector of $\overline{AB}$. None of this knowledge or vocabulary is expectable at this level, and so "in exactly the same direction" is as much precision as can probably be expected.

3. If Sam is right, and we can "stretch" point $C$, are there any points on the board that could be point $C$? If so, how would you prove that the point you found *is* equidistant from $A$ and $B$?

   
   
   Figure 6

   The two points found in lines 15 and 16 define a line. All points on this line (except for one!) can be the third vertex of an isosceles triangle whose other two vertices are $A$ and $B$. Some points on this line are also lattice points. One of those lattice points appears also to be on this geoboard. This is point $C$ in Figure 6. It looks right, but to be sure, we now need to verify that the distances to $A$ and $B$ are the same. We can measure $\overline{AC}$ as 5 units by counting. We can use the students' up-and-over method to count 4 up and 3 over from point $B$ to point $C$. This creates a right triangle with legs 3 and 4, and we could use the Pythagorean Theorem to show that the hypotenuse ($\overline{BC}$) is length 5, which is what we need it to be.

4. How are the students using structure (MP 7) in this dialogue to support their conjectures?

    Refer to the Mathematical Overview section for notes related to this question.

5. Why does the up-and-over strategy work as a proof?

    Refer to the Mathematical Overview section (MP 7 and Commentary on the Mathematics) for notes related to this question.

6. If we remove the constraint of using only the pegs on the geoboard, where are all the points that would create an isosceles triangle with $\overline{AB}$?

    The students have already identified a portion of this set of points when they "stretched" point $C$ and found a line on which all of these points could lie (see the discussion of Teacher Reflection Question 2). At this point, students might recognize this line as one that makes right angles with $\overline{AB}$ and that it cuts $\overline{AB}$ in half. In high school mathematics, this will be called the "perpendicular bisector of $\overline{AB}$." The remainder of the points that satisfy this lie on two circles centered at $A$ and $B$ with a radius the same as the length of $\overline{AB}$. Of course, on each circle, there will be one point that does not make a triangle.

***Possible Responses to Student Discussion Questions***

1. Discuss why the up-and-over technique works to prove that the lengths are the same.

    In moving "up and over," the students in this dialogue are essentially creating rectangles with sides of 3 and 1 (or sides of 2 and 1). Since rectangles with the same dimensions are the "same size," then the diagonals across them are also the same size. These diagonals are the sides of the isosceles triangles the students are finding.

2. Have the students in the dialogue found all of the isosceles triangles possible?

    The short answer is "almost." The students have considered where the third vertex could be for all possible pairs of congruent sides ($\overline{AB}$ and $\overline{AC}$, $\overline{AB}$ and $\overline{BC}$, and $\overline{AC}$ and $\overline{BC}$). When working to make $\overline{AB}$ and $\overline{AC}$ congruent, they name all three vertices that fall within the geoboard grid. When working to make $\overline{AB}$ and $\overline{BC}$ congruent, they name two of the three vertices that fall within the geoboard grid, but miss the vertex that is "up one and left three" from point $B$. And when working to make $\overline{AC}$ and $\overline{BC}$ congruent, they find two vertices within the geoboard grid that work, and explain a strategy for "stretching" the triangle to find others. "Stretching" the triangle by moving the third vertex along the line that is equidistant from points $A$ and $B$ would locate one additional grid point on this geoboard, though the students do not actually name which point that is during the dialogue.

3. What does Sam mean in line 17 by "stretch out point $C$"? If you did what Sam means, are there any points on the geoboard that you could reach exactly?

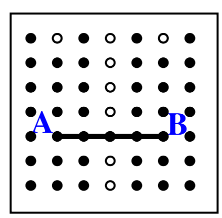
   Sam knows that points don't "stretch." Sam is really describing the idea of stretching (or shrinking) the distance from $C$ to $\overline{AB}$ by moving point $C$ so that sides $\overline{AC}$ and $\overline{BC}$ remain equal to each other but are longer or shorter than they are now. If we draw a line connecting the two points they already found that work, point $C$ can be placed anywhere along that line to make $\overline{AC}$ the same length as $\overline{BC}$. If we extend that line in both directions, we can see one more place on the geoboard where it looks like we can place point $C$. (See Figure 6).

***Possible Responses to Related Mathematics Tasks***

1. How would the results of the original problem change if you were allowed to put more than one geoboard together?

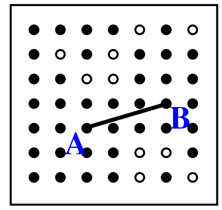
   Presuming that we decide to use geoboards of the same size and shape, "attaching" these to the original would allow for more positions for point $C$. Using point $A$ as the starting position, the up-1-and-over-3 method would allow for eight total possible locations (up or down, left or right, 1 or 3) of point $C$ that would cause $\overline{AC}$ to be the same length as $\overline{AB}$. One of those possible locations is point $B$. Another of those positions would be collinear with $\overline{AB}$ so it would not form a triangle. Thus, there are 6 locations on this configuration for point $C$ that would make $\overline{AC}$ the same length as $\overline{AB}$. The same reasoning leads to the fact that there are also six locations for point $C$ that would cause $\overline{BC}$ to be the same length as $\overline{AB}$. Finally, depending on how many additional geoboards are added, any points (except the midpoint of $\overline{AB}$) that are on the line hinted at by the students in line 17 of the dialogue would cause $\overline{AC}$ to be the same length as $\overline{BC}$.

2. If you are free to move point $A$ (leaving point $B$ where it is), where would you move $A$ to maximize the number of possible locations for point $C$ on the geoboard? What if you could move point $B$ (leave point $A$)?

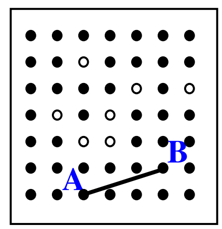
   If we move point $A$ up 1 peg and then move it left 1 peg, we create a segment 4 units long. This creates a column of pegs that could all (except one, of course) be locations for point $C$ (such that $\overline{AC}$ and $\overline{BC}$ are the same length). The other two possibilities are shown that cause $\overline{AB}$ and $\overline{AC}$ (or $\overline{AB}$ and $\overline{BC}$) to be the same length. The same number of possibilities exists if we place point $A$ so that $\overline{AB}$ is 2 units long, though the actual positions of the points will be different. However, if we stretch $\overline{AB}$ to 6 units long, we eliminate the possibility that $\overline{AB}$ and $\overline{AC}$ (or $\overline{AB}$ and $\overline{BC}$) could be the same length. If the length of $\overline{AB}$ is an odd number, we remove all the "centerline" possibilities because that midline no longer contains lattice points. This same reasoning applies to moving point $B$.

3. Let's say that you can move both points, but had to leave them exactly the same distance from each other as they are now. In what position(s) can you place them to *maximize* the number of isosceles triangles that can be made?




Through lots of exploration, we discover that moving $\overline{AB}$ to a position that is as close to center as possible will maximize the number of possible locations for point $C$ to make an isosceles triangle. In the figure to the right, the white dots represent these possibilities. You can see that this is only a shift up one row from the original position of $\overline{AB}$. In doing this, we lose one possibility in the top row, but we gain two in the bottom row. If we move $\overline{AB}$ up another row, we gain two more possibilities in the lower left, but we lose two in the upper right. If we move the segment right, we gain two locations, but we also lose two (top and bottom right corners). We can consider any rotations of $\overline{AB}$ by considering the rotational symmetry of the board and the current layout. This will show that any rotation nets the same results.

4. In what position(s) can you place $\overline{AB}$ to *minimize* the number of isosceles triangles that can be made?




By paying attention to the explorations in maximizing the number of possibilities, we can answer this question at the same time. In this case, if we take $\overline{AB}$ to an extreme position (a corner or an edge), we can make a net change of eliminating possibilities. In the figure at the right, again, the white dots represent the possible positions for point $C$. In this case, being in the center of the edge is more limiting because if we move toward either right or left corner, we eliminate some possibilities but we gain others and the net result is either a gain or zero change.

5. Imagine the exact same problem posed on another geoboard of the exact same area, but not the exact same shape. How would that affect the number of possible isosceles triangles?

This opens up all sorts of possibilities. Some geoboard shapes to explore could be triangles, non-square rectangles, and circles.

6. How does the problem change if we want an equilateral triangle instead of isosceles? What if we want to guarantee a scalene triangle?

Again, through some thorough exploration, we can discover that an equilateral triangle is not possible on a geoboard. At this level, it's enough that students know and believe this intuitively. However, it can be proven. If a line contains two lattice points, the slope of this line is rational (that is, the rise and run are both integers). Using either of these lattice points as a pivot, rotating the line $60^\circ$ (the measure of the angle between any two sides of an equilateral triangle) creates a new line with an irrational slope. (The proof of this is left to the reader to explore.) This irrational slope inherently eliminates any other lattice points on the line. This helps us with the second part of this question as well. If a triangle is not equilateral or isosceles, it must be scalene. So, any point on the geoboard that does not create an isosceles triangle will create a scalene triangle. It's important to note that if we move $\overline{AB}$ to the right or left, there will be one point that is collinear with both points, and this point would therefore not create ANY form of a triangle.