Making Sense of a Quadratic Function

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Making Sense of a Quadratic Function* Illustration: This Illustration's student dialogue shows the conversation among three students who are asked if they can find a quadratic, f, such that $f(1) = f(2) = f(3)$ or $f(f(1)) = f(f(2)) = f(f(3))$. While working on a solution students use graphs of parabolas and reason about what inputs and outputs would have to be in order to meet the conditions of the problem.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 2: Reason abstractly and quantitatively.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 10–12

Target Content Domain: Interpreting Functions (Functions Conceptual Category)

Highlighted Standard(s) for Mathematical Content

HSF-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

HSF-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Math Topic Keywords: algebra, quadratic, functions, function notation, composition, graphing, parabola

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Making Sense of a Quadratic Function

Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

For a quadratic function $f(x) = ax^2 + bx + c$, determine whether or not it is possible that $f(1) = f(2) = f(3)$.

Determine all of the quadratic functions $f(x) = ax^2 + bx + c$ that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$.

Task Source: Adapted from Problem from Mortici, C. *600 de Probleme*. Zalău: Gil. 2001.

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students are familiar with function notation, composition of functions, quadratic functions and parabolas.[1]

(1) Chris: Come on!! How are we supposed to do this? We've *never* been taught *anything* like it! I don't even know what f f f f, or whatever, *means*.

(2) Lee: Yeah, we've never even *seen* anything like this.

(3) Matei: Well, maybe we can figure it out. The first question looks easier. Let's start there. We have to decide if $f(1) = f(2) = f(3)$ can be a true statement.

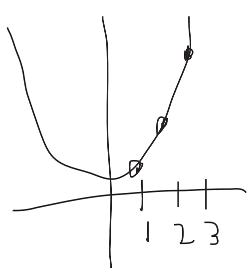
(4) Chris: Ok, but how can we show it? We don't even have a specific function, like $f(x) = 2x^2 + 2x + 4$. If we did, we could just plug in the values and evaluate, but here, we have nothing!

(5) Matei: Well, we know the function is quadratic…

[Students pause to think.]

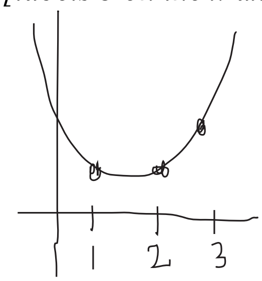
(6) Lee: So its graph looks something like this. *[He sketches a rough parabola and draws axes so that the parabola is symmetric about the y-axis.]*

(7) Chris: So when x equals 1, 2, and 3, the values of f are here, here, and here. *[labels 1, 2, 3 on the x-axis and draws dots on the graph corresponding to those inputs]* None of the outputs are equal…

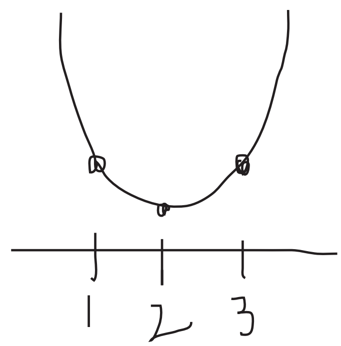


[1] In this Illustration, the word *parabola* is used as a verbal description for the shape of the graph of a quadratic function. This might seem different from the definition of parabola as the locus of points that is equidistant from a point (the focus) and a line (the directrix). In fact, these two are the same. (For more on this, see *CME Project* Algebra 2.)

(8) Lee: But they could be! Well, at least two of them could be. Look, what if $x = 1$ and $x = 2$ are here and here? *[draws a new parabola and labels 1 and 2 on the x-axis as below so that their outputs are equal]* Then $f(1) = f(2)$. But $x = 3$ would have to be out here somewhere. *[labels 3 on the x-axis]* So $f(3)$ would be different.



(9) Chris: Oh, right. But maybe we can do better. We could make $f(1)$ and $f(3)$ the same… *[draws picture]* …but then the picture would look like this, and $f(2)$ would be different.



Or $f(2) = f(3)$… Oh, of course! For any quadratic equation, a max of two inputs can give the same output, but never three.

(10) Matei: So what about the second part of the question?

[They sit and think for a few minutes.]

(11) Lee: I wonder what the graph of $y = f(f(x))$ would look like. We know $f(x)$ looks like a parabola. Then $f(f(x))$ would be… Well, if $f(x)$ equaled x^2, then $f(f(x))$ would be $(x^2)^2$, which is x^4. So *that* graph… *[trails off]*

(12) Chris: Instead of talking about what happens to x, why don't we talk about the specific numbers they give us? We know $f(1)$ is the output on the graph of $f(x)$ when $x = 1$. Why don't we make up a number for $f(1)$? What if we pretend $f(1)$ is, I don't know, 20?

(13) Lee: You're right. $f(1)$ and $f(2)$ and $f(3)$ are outputs of f, so they're *numbers*. So if we pretend that the numbers are, like, 20, 21, and 22, then that complicated f of f of 1, well, you know, *that* thing *[points to $f(f(1)) = f(f(2)) = f(f(3))$]* is like $f(20) = f(21) = f(22)$.

(14) Chris: Right. So it's like the first problem again?

[silence]

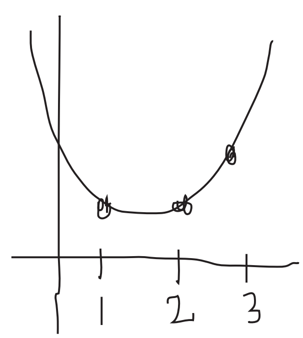
(15) Matei: This is definitely connected to the first problem.

(16) Chris: Lee, in your example, you said we can think of that complicated thing like $f(20) = f(21) = f(22)$. But we already know that's impossible from the first problem we did; since the graph of f is a parabola, only two of those can be equal. So is the answer that there are no functions that satisfy the relationship?

(17) Matei: No, I think it may be possible.

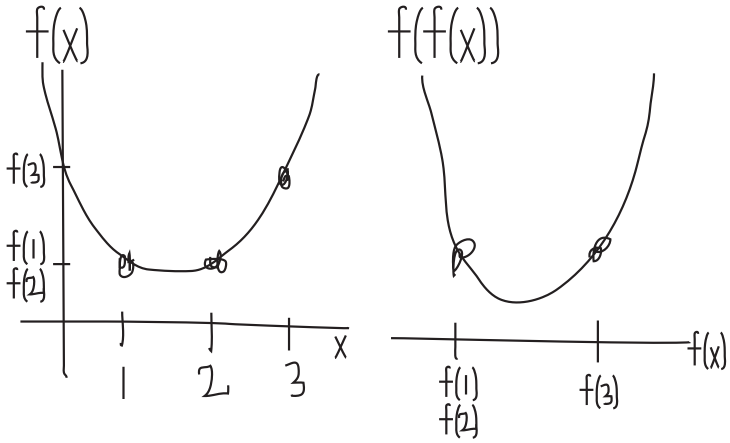
(18) Chris: What? We already showed that if the three inputs are different, the three outputs can't all be the same!

(19) Matei: You're right. But what if the inputs aren't all different? Look at Lee's drawing again. *[points to the sketch below]* What if $f(1) = f(2)$? Let's say it *is* 20, for example. Then that complicated thing actually says $f(20) = f(20) = f(22)$ or something. That's possible… Do you see it?



(20) Lee: You're saying that $f(1)$ could equal 20 and $f(2)$ could *also* equal 20 and $f(3)$ could be something else. Oh yeah, that could happen. So…

(21) Matei: So if $f(1) = f(2)$, then $f(f(1)) = f(f(2))$. If the inputs are the same, the outputs would have to be the same, too. And then $f(3)$, we know, is *different* from $f(1)$ and $f(2)$. Hmm… *[They all think.]* But then we're trying to see if $f(f(3))$ can still give the same output… *[They all think again.]* So it's like we start with this graph *[draws first parabola]* and then plug in the outputs again to get a graph that looks like this. *[draws second parabola, putting values of $f(x)$ on the horizontal axis]* This is only a sketch, but do you see what I mean?



(22) Chris: So, wait. Are you saying that the only way for $f(f(1))$ and $f(f(2))$ and $f(f(3))$ to be equal is… Oh, sure! If two of them have the exact same input—like if $f(1) = f(2)$—then their outputs $f(f(1)) = f(f(2))$ are *guaranteed* equal. Then all we need is…

(23) Lee: Right, so, we're supposing that $f(1)$ and $f(2)$ both equal some number we'll call v. Even though $f(3)$ *must* be a different number—let's call that w—we're OK, because it *is* possible for $f(v)$ to equal $f(w)$ for at least some values of v and w. Two different inputs *can* have the same output! Yay!! We found it!

(24) Chris: Wow, I didn't think we could do this problem, but I think we almost have it. Can we work out the few remaining details?

Making Sense of a Quadratic Function

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of the students in the dialogue engaging in the Standards for Mathematical Practice?

2. In what ways is this task difficult? Where do you anticipate students having trouble with this task?

3. What were the critical steps or ways of thinking/acting or insights that moved the students from "this is crazy impossible" to "we have an idea that might work"?

4. The first part of the task asks students to consider a situation and determine "is this possible?" Consider the content you are currently teaching to your own students, and think of "Is it possible?" questions you might ask in that context.

5. In line 11, Lee realizes that the function $f(f(x))$ will involve an x^4 term and so trails off in an effort to describe the graph of $f(f(x))$. But then in line 21, Matei draws a graph for $f(f(x))$, and seems to claim that the graph of $f(f(x))$ is a parabola. Describe how both of these statements can be mathematically correct.

6. The students in this Illustration use sketches of parabolas to approach this problem. What other approaches would you expect to see from students?

7. What have students figured out by the end of the Student Dialogue? In what ways does this task deepen student understanding of function composition?

8. The task is to determine all of the quadratic functions that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$. Predict the form of the solution. Is there only one possible function? Infinitely many possible functions? What *can* you say about the solution?

9. Have you worked out the problem yet? How should the students proceed now that they understand the problem and can imagine the form that the solution will take? Determine all of the quadratic functions of the form $f(x) = ax^2 + bx + c$ that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$.

Making Sense of a Quadratic Function

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	The Student Dialogue begins with students expressing confusion about the problem (lines 1–2), saying, "We've *never* been taught *anything* like it." The students make inroads into the problem by shifting their focus from what they don't know to what they do know. Matei starts in line 5 by stating, "we know the function is quadratic…" and the students proceed from there. The students have found an "entry point" that they use to "explain to themselves the meaning of [the] problem." The task itself includes a simpler part and more complex one. The students tackle the simpler problem first, and then starting in line 11, they look for similarities between the simpler problem and the more complex task. Throughout this Illustration, the students also demonstrate this standard by spending time "making conjectures about the form and meaning of the solution… rather than simply jumping into a solution attempt." By the end of the Student Dialogue, the students have figured out that there are three cases for them to consider, and are now ready to begin working out the details.
 Reason abstractly and quantitatively.	Students use quantitative reasoning, which "entails habits of… attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects." For example, Chris demonstrates an understanding of how to compute $f(1)$ for a specific function (line 4) and also that $f(1)$ refers to the output of a function corresponding to an input of 1 (line 7). The task itself leads the students to reason abstractly (by asking them to consider a general quadratic of the form $ax^2 + bx + c$) and quantitatively (by asking about such a function at the specific values of 1, 2, and 3). The students move back and forth between the abstract and concrete in the Student Dialogue. For example, to talk about the meaning of $f(f(1))$, Lee gives a concrete example: "Pretend that [$f(1)$, $f(2)$, and $f(3)$] are, like, 20, 21, and 22" (line 13). The students use these concrete numbers to talk about the problem (lines 16, 19, 20), and then Matei uses the ideas from their conversation to represent the problem abstractly (line 21), drawing a graph with the labels $f(1)$, $f(2)$, and $f(3)$, rather than the concrete numbers they had been using.

Making Sense of a Quadratic Function

 Use appropriate tools strategically.	Strictly speaking, graphs are not required to solve this problem, but it is hard to imagine the conversation students would have been able to have without them. Using graphs to illustrate and develop their ideas helps the students form a common understanding. Drawing a parabola (lines 6–8) helps them make their first breakthrough to making sense of the problem. Thinking of graphs also helps the students understand function composition. In line 11, Lee trails off after thinking of how they might use graphs as a tool to make sense of $f(f(x))$ since it would include an x^4 term. Then, the students develop the idea that they can think of $f(f(x))$ as "f of something." Matei uses this understanding to draw the two graphs of $f(x)$ and $f(f(x))$ side by side (line 21) as two parabolas. Further examination of Matei's drawing can lead to the discovery of several subtle points. First, the labels on the axes are very important in this picture because the first graph is a graph of f with respect to x whereas the second graph is of f with respect to $f(x)$. Note that the graph of $f(f(x))$ with respect to x would *not* be a parabola, but would be, as Lee noticed, some quartic function. Second, the two side-by-side drawings are actually of the same exact parabola since they are both graphs of the function f and this constrains the possibility for the possible values that a, b, and c may take in the quadratic function f. Note that the original task does not mention using graphs. The students in the Student Dialogue demonstrate this standard when they "make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations." In considering what, if anything, they knew about quadratic equations, the students realized that one tool they had was knowing that the graph of a quadratic looks like a parabola, and they are able to use this tool effectively to gain insight into the problem.
 Look for and make use of structure	Students who demonstrate MP 7 "step back for an overview and shift perspective" and "see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects." In this Illustration, students demonstrate this standard most clearly beginning in line 11 as they move from considering the first part of the task to the second. They have just found that it is not possible that $f(1) = f(2) = f(3)$, though it is possible for two of these at a time to be equal. Then the students notice (beginning in line 13) that the two parts of the task have a similar structure. Considering the relationship $f(f(1)) = f(f(2)) = f(f(3))$ is "like the first problem again" if you consider that $f(1)$, $f(2)$, and $f(3)$ are all outputs of $f(x)$ and, therefore, can be thought of as numbers. Lee compares $f(f(1)) = f(f(2)) = f(f(3))$ to the statement $f(20) = f(21) = f(22)$, drawing attention to the similarity in structure between this part of the task and the work they have already done. Then in lines 16–20, the students further clarify ways in which the second task is and isn't like the first. For example, Matei notices (line 19) that it is possible for $f(f(1)) = f(f(2)) = f(f(3))$ to translate to the statement $f(20) = f(20) = f(22)$, which leads the students to conclude that it is possible for a quadratic function to satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$.

Making Sense of a Quadratic Function

Commentary on the Mathematics

What follows is a solution to the mathematical task. It is quite possible to explore this mathematical task without ever getting to the solution. The interesting mathematical thinking occurs in making sense of the problem rather than in the algebraic manipulation that follows.

But here's a partial working out of the solution, for those who are interested.

There are three cases. Case 1: $f(1) = f(2)$. Case 2: $f(2) = f(3)$. Case 3: $f(1) = f(3)$.

Case 1: $f(1) = f(2)$

$f(1) = a(1)^2 + b(1) + c = a + b + c$
$f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$
$f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$

Since $f(1) = f(2)$: $a + b + c = 4a + 2b + c$, and therefore $b = –3a$

Rewrite $f(1)$, $f(2)$, and $f(3)$ using $b = –3a$. Note that $f(1) = f(2)$, as expected.

$f(1) = a + b + c = a – 3a + c = –2a + c$
$f(2) = 4a + 2b + c = 4a – 6a + c = –2a + c$
$f(3) = 9a + 3b + c = 9a – 9a + c = c$

So $f(f(1)) = f(f(2)) = a(–2a + c)^2 – 3a(–2a + c) + c$
$= a(4a^2 – 4ac + c^2) + 6a^2 – 3ac + c$
$= 4a^3 – 4a^2c + ac^2 + 6a^2 – 3ac + c$

And $f(f(3)) = ac^2 – 3ac + c$

Let $f(f(1)) = f(f(2)) = f(f(3))$:

$4a^3 – 4a^2c + ac^2 + 6a^2 – 3ac + c = ac^2 – 3ac + c$
$4a^3 – 4a^2c + 6a^2 = 0$
$2a^2(2a – 2c + 3) = 0$

Since $a \neq 0$ (because the function must be quadratic), $2a – 2c + 3 = 0$.

Case 1 leads us to the family of quadratic functions in which $b = –3a$ and $2a – 2c + 3 = 0$.

In a similar fashion, we find that Case 2 leads to the family of quadratic functions in which $b = –5a$ and $10a – 2c + 5 = 0$, and Case 3 leads to the family of quadratic functions in which $b = –4a$ and $7a – 2c + 4 = 0$.

Alternatively, we can make a more general case from this. We're still considering the quadratic function $f(x)$ in which $f(f(1)) = f(f(2)) = f(f(3))$. Since it continues to be true that only two $f(x)$ can be equal, we will explore the case where $f(1) = f(2)$ and name that value r. Then $f(x) – r$ has zeroes at 1 and 2, so it is divisible by $(x – 1)$ and $(x – 2)$ and also by their product. Because f is quadratic, these must be all of the linear factors and hence there is a number $k \neq 0$ so that:

$$f(x) - r = k(x-1)(x-2)$$

$$f(x) = k(x-1)(x-2) + r$$

From this we calculate that $f(3) = 2k + r$. Hence,

$$f(f(1)) = f(r) = k(r-1)(r-2) + r$$
$$f(f(2)) = f(r) = k(r-1)(r-2) + r$$
$$f(f(3)) = f(2k+r) = k(2k+r-1)(2k+r-2) + r$$

In the problem, these are equal, so

$$k(r-1)(r-2) + r = k(2k+r-1)(2k+r-2) + r$$

subtracting r and dividing by k (reiterating $k \neq 0$)

$$(r-1)(r-2) = (2k+r-1)(2k+r-2)$$

expanding:

$$r^2 - 3r + 2 = (2k+r)^2 - 3(2k+r) + 2$$

through more manipulation

$$4k^2 + 4kr - 6k = 0$$

and finally factoring out a $2k$

$$2k(2k+2r-3) = 0$$

Since $k \neq 0$, then $2k + 2r - 3 = 0$, which leads to

$$r = \frac{3-2k}{2}$$
$$f(x) = kx^2 - 3kx + \frac{2k+3}{2}$$

Any quadratic, then, in this form will meet the conditions of $f(1) = f(2)$ and $f(f(1)) = f(f(2)) = f(f(3))$. In similar fashion, we can explore the other two cases and get the following:

$$f(x) = kx^2 - 5kx + \frac{10k+5}{2} \quad \text{for Case 2: } f(2)=f(3)$$

and

$$f(x) = kx^2 - 4kx + \frac{7k+4}{2} \quad \text{for Case 3: } f(1) = f(3)$$

Through comparison of the structure of these equations with those results shown previously, we can show that they are the same. This process can be generalized one more level to find a function in which $f(f(m)) = f(f(n)) = f(f(p))$ for any values m, n and p, but we will leave that to you.

Making Sense of a Quadratic Function

Evidence of the Content Standards

Both identified content standards come from the High School: Functions (Interpreting Functions) domain. In this Illustration, we see that students “use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context” (HSF-IF.A.2). The students are able to explain why it is not possible that $f(1) = f(2) = f(3)$ in the context of a quadratic function $f(x) = ax^2 + bx + c$, but why it is possible that $f(f(1)) = f(f(2)) = f(f(3))$. The students approach the problem graphically, and thereby “graph functions expressed symbolically and show key features of the graph, by hand…” (HSF-IF.C.7) as they start with a generic parabola (line 6) and add key features such as labels along the input axis (line 9) and output axis (line 21) to support their explanations.

Making Sense of a Quadratic Function

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. From the Student Dialogue, we know that it is not possible that $f(1) = f(2) = f(3)$ when f is a quadratic function. Draw or write an expression for a function f (not quadratic) for which $f(1) = f(2) = f(3)$.

2. To determine whether it is possible that $f(1) = f(2) = f(3)$, the students sketch several graphs. As with any sketch, some aspects are more accurate than others, and there's a danger that the sketch oversimplifies some aspects that turn out to be important.
   (a) Parabolas may be concave up (with the vertex at the minimum) or concave down (with the vertex at the maximum). The students in the Student Dialogue consider only parabolas that are concave up. Does the direction matter for their argument? Explain.
   (b) In line 9, Chris draws a parabola to show that $f(1)$ and $f(3)$ can be the same. In the drawing, it looks like $(2, f(2))$ is the vertex. Is this aspect of the drawing accurate?
   (c) In line 8, Lee draws a parabola so that $f(1) = f(2)$. From the drawing, it looks like $f(3)$ has the same value as the y-intercept of the parabola. Is this aspect of the drawing accurate?

3. In line 11, Lee realizes that the function $f(f(x))$ will involve an x^4 term and so trails off in an effort to describe the graph of $f(f(x))$. But then in line 21, Matei draws a graph for $f(f(x))$, and seems to claim that the graph of $f(f(x))$ is a parabola. Describe how both of these statements can be mathematically correct.

4. In your own words, interpret the drawing in line 21. Then make similar diagrams for when $f(1) = f(3)$ and for when $f(2) = f(3)$.

5. The second part of the task is to determine *all* of the quadratic functions that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$. Predict the number of solutions there are. Does *any* quadratic function work?

6. Finishing this task requires a lot of algebraic manipulation. If you are already well on your way towards a solution, keep going! If you would like a little guidance along the way, the following questions may help.
   (a) Find $f(1)$, $f(2)$, and $f(3)$ in terms of a, b, and c.
   (b) If $f(1) = f(2)$, then that constrains the relationship between a and b. Use that constraint to rewrite $f(1)$, $f(2)$, and $f(3)$.
   (c) Find $f(f(1))$, $f(f(2))$, and $f(f(3))$ in terms of a and c. (Remember, you can write b in terms of a.) Finally, figure out how a, b, and c must be related for $f(f(1)) = f(f(2)) = f(f(3))$.
   (d) Repeat the problem for the case where $f(1) = f(3)$.
   (e) And again for the case where $f(2) = f(3)$.
   (f) State your solution to the task in a way that makes it clear you know what you've found as a result of all of that algebra.

Related Mathematics Tasks

1. For a quadratic function of the form $f(x) = ax^2 + bx + c$, determine whether or not it's possible that $f(1) = f(3)$ and $f(2) = f(4)$.

2. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$, and let $g(x)$ be a linear function of the form $g(x) = px + q$. Is it possible that $f(g(1)) = f(g(2)) = f(g(3))$? Explain.

3. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$, and let $g(x)$ be a linear function of the form $g(x) = px + q$. Is it possible that $g(f(1)) = g(f(2)) = g(f(3))$? Explain.

4. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$. Is it possible to find p, q, r, and s such that $f(f(p)) = f(f(q)) = f(f(r)) = f(f(s))$? Explain.

5. In this Illustration, students make sense of $f(f(x))$ by seeing that they can think of the structure as a "problem-within-a-problem." This kind of reasoning shows up in mathematics surprisingly often. Here is another task that you might approach by taking advantage of its problem-within-a-problem structure:

$$\text{Find } \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$$

Making Sense of a Quadratic Function

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of the students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. In what ways is this task difficult? Where do you anticipate students having trouble with this task?

   This challenging task requires no knowledge beyond typical high school content. The challenge comes from requiring students to think not about the nature of a particular function (see line 4), but about the characteristics of an entire class of functions: quadratics. The notation is unfamiliar, and as Chris points out in line 1, it may be that students have "never been taught anything like it." The answer is not a specific number, nor can they rely on numbers to solve it, nor is the answer a specific function; rather, the answer is *all* of the possible quadratic functions that satisfy the given relationship.

3. What were the critical steps or ways of thinking/acting or insights that moved the students from "this is crazy impossible" to "we have an idea that might work"?

   In line 3, Matei sensibly ignores the more complicated second half of the task and focuses on the simpler first half. When Chris points out (line 4) that even this simpler task is difficult, Matei takes another step back and asks the group to consider what they do know by pointing out that they know the function is quadratic (line 5). Lee replies with a piece of knowledge common to the students—that the graph of a quadratic has a familiar shape (line 6). This initial process of stepping back to think about what they already know that might help them with this new problem is a key step that leads to an approach. Another critical step occurs in lines 13–16 as the students look at the second half of the task and look for connections to the first half. Students look for ways that their previous reasoning can be extended to a new, related situation.

4. The first part of the task asks students to consider a situation and determine "is this possible?" Consider the content you are currently teaching to your own students, and think of "Is it possible?" questions you might ask in that context.

   Discuss with colleagues to share ideas. Such questions can provide opportunities for students to demonstrate MP 3 (construct viable arguments and critique the reasoning of others). Even though this Standard for Mathematical Practice was not explicitly highlighted in this Illustration, we see that the students build a mathematical argument to show that it is not possible for $f(1) = f(2) = f(3)$ for a quadratic function f. Other examples of "Is it possible?" questions about functions include:

- Is it possible for $ab = -ab$ for real numbers a and b?
- Is it possible for $-c > 0$ for a real number c?
- Is it possible for $f(a) = f(-a)$ for any function f? For a linear function f?
- Is it possible to find x so that $x + 2 = x - 2$?
- Is it possible to find a function f so that $f(x) + 2 = f(x) - 2$?
- Is it possible to find a function f so that $f(x + 2) = f(x - 2)$?
- Is it possible for a function f to satisfy $f(f(x)) = x$ for all real numbers x?

5. In line 11, Lee realizes that the function $f(f(x))$ will involve an x^4 term and so trails off in an effort to describe the graph of $f(f(x))$. But then in line 21, Matei draws a graph for $f(f(x))$, and seems to claim that the graph of $f(f(x))$ is a parabola. Describe how both of these statements can be mathematically correct.

   Lee is correct to say that $f(f(x))$ will be a quartic function, so the graph of $f(f(x))$ is not a parabola. But Matei is also correct in line 21 when he draws a parabola for the graph of $f(f(x))$ because the input in his graph is not x, but is $f(x)$. In other words, his two graphs are the graphs of $f(x)$ and $f(y)$, where $y = f(x)$. And since f represents the same quadratic function, Matei is correct in drawing two parabolas (in fact, the same parabola) to first represent f as a function of x, and then to represent f as a function of $f(x)$.

6. The students in this Illustration use sketches of parabolas to approach this problem. What other approaches would you expect to see from students?

   Instead of examining the graph of a quadratic function, students might use the numbers provided to approach the problem. Doing this, they would find that $f(1) = a + b + c$, $f(2) = 4a + 2b + c$, and $f(3) = 9a + 3b + c$. Since $f(1) = f(2) = f(3)$, students would find that $a + b = 4a + 2b = 9a + 3b$, which is only satisfied when $a = b = 0$. But if $a = b = 0$, then the function is no longer quadratic, so the students would have shown that it is not possible that $f(1) = f(2) = f(3)$. Students can take this analysis further to address the second half of the task. This analysis is shown in depth in the Mathematical Overview.

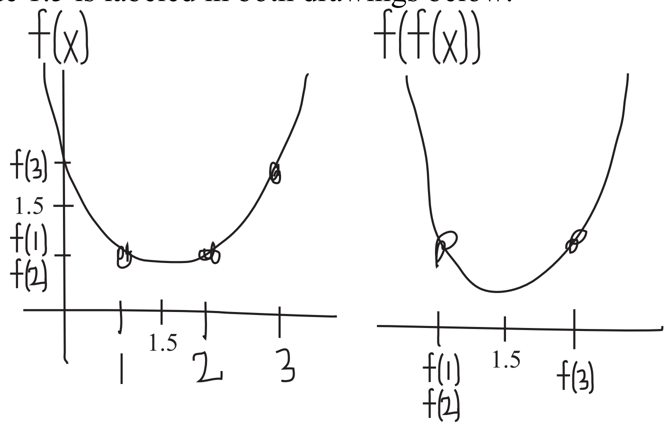
7. What have students figured out by the end of the Student Dialogue? In what ways does this task deepen student understanding of function composition?

   By the end of the Student Dialogue, the students have figured out that there are three possible ways for a function of the form $f(x) = ax^2 + bx + c$ to satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$. One of the following must be true: $f(1) = f(2)$, $f(2) = f(3)$, *or* $f(1) = f(3)$. They reached this conclusion by realizing that a quadratic function can produce the same output for at most two distinct inputs. Students are deepening their understanding of function composition when Chris and Lee (lines 12–13) realize that they can make up a number for $f(1)$ because $f(1)$, $f(2)$, and $f(3)$ are numbers, which become the inputs in the outer function. Matei's drawing in line 21 shows that the students have figured out how the output $f(x)$ becomes the input to f in the expression $f(f(x))$ by putting values of $f(x)$ on the horizontal axis.

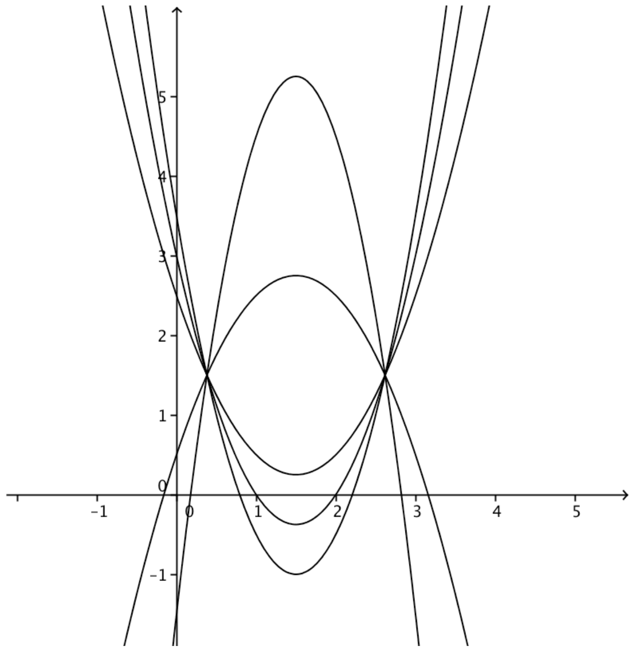
What they learn is not only about composition. To solve this problem, they had to reconstruct what they know about quadratics. They are learning familiar content "inside out and backward": instead of solving for x (a single number), or even a, b, and c (specifying a single function), they are solving for a relationship among a, b, and c: a family of functions.

8. The task is to determine all of the quadratic functions that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$. Predict the form of the solution. Is there only one possible function? Infinitely many possible functions? What *can* you say about the solution?

There are three different cases to consider: when $f(1) = f(2)$, $f(2) = f(3)$, and $f(1) = f(3)$. In each case, a family of functions will satisfy the relationship. For example, consider the case when $f(1) = f(2)$, shown by Matei's drawing in line 21. Because of the symmetry of a parabola, we can fill in some more details to help us visualize the possible parabolas that would satisfy the relationship. Because $f(1) = f(2)$, we know that the vertex of the graph of $f(x)$ occurs at $(1.5, f(1.5))$. That also means that 1.5 is the average of $f(f(1)) = f(f(2))$ and $f(f(3))$ because the two drawings are of the same function, f (note that in the second case, the graph is still a picture of $f(x)$, but where the input values are themselves values of $f(x)$). The value 1.5 is labeled in both drawings below.



From this picture, it is possible to see that a family of parabolas can be made to fit these constraints, all with a vertex at $(1.5, f(1.5))$, and for which the average of $f(1)$ and $f(3)$ is also 1.5. A few of these parabolas are shown below.



A similar family of quadratic functions can be constructed for each of the two other cases where $f(2) = f(3)$ *and* $f(1) = f(3)$. So we expect the solution to be three families of quadratic functions.

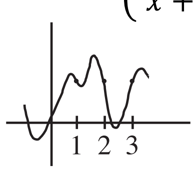
9. Have you worked out the problem yet? How should the students proceed now that they understand the problem and can imagine the form that the solution will take? Determine all of the quadratic functions of the form $f(x) = ax^2 + bx + c$ that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$.

   The solution is shown in the Mathematical Overview. To proceed, students should look for what they can determine by plugging in 1, 2, and 3 for x, using the relationships they have identified as additional constraints.

Possible Responses to Student Discussion Questions

1. From the Student Dialogue, we know that it is not possible that $f(1) = f(2) = f(3)$ when f is a quadratic function. Draw or write an expression for a function f (not quadratic) for which $f(1) = f(2) = f(3)$.

   There are many possible responses, including:

   - $f(x) = 5$
   - $f(x) = 5(x - 1)(x - 2)(x - 3) + 17$
   - $f(x) = \begin{cases} 2x + 3 \; if \; x \leq 1 \\ 5 \; if \; 1 < x \leq 2 \\ x + 2 \; if \; x > 2 \end{cases}$
   - 

2. To determine whether it is possible that $f(1) = f(2) = f(3)$, the students sketch several graphs. As with any sketch, some aspects are more accurate than others, and there's a danger that the sketch oversimplifies some aspects that turn out to be important.
   (a) Parabolas may be concave up (with the vertex at the minimum) or concave down (with the vertex at the maximum). The students in the Student Dialogue consider only parabolas that are concave up. Does the direction matter for their argument? Explain.
   (b) In line 9, Chris draws a parabola to show that $f(1)$ and $f(3)$ can be the same. In the drawing, it looks like $(2, f(2))$ is the vertex. Is this aspect of the drawing accurate?
   (c) In line 8, Lee draws a parabola so that $f(1) = f(2)$. From the drawing, it looks like $f(3)$ has the same value as the y-intercept of the parabola. Is this aspect of the drawing accurate?

   (a) The same arguments can be made with parabolas that are concave down. Students should be able to draw concave down parabolas for which $f(1) = f(2)$, $f(1) = f(3)$, and $f(2) = f(3)$.
   (b) It is correct that $(2, f(2))$ is the vertex of this parabola. This must be true because parabolas are symmetric.
   (c) It is true that $f(3) = f(0)$, again because parabolas are symmetric.

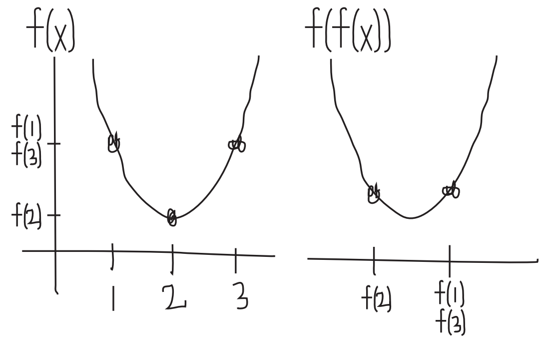
3. In line 11, Lee realizes that the function $f(f(x))$ will involve an x^4 term and so trails off in an effort to describe the graph of $f(f(x))$. But then in line 21, Matei draws a graph for $f(f(x))$, and seems to claim that the graph of $f(f(x))$ is a parabola. Describe how both of these statements can be mathematically correct.

   Lee is correct to say that $f(f(x))$ will be a quartic function, so the graph of $f(f(x))$ is not a parabola. But Matei is also correct in line 21 when he draws a parabola for the graph of $f(f(x))$ because the input in his graph is not x, but is $f(x)$. In other words, his two graphs are the graphs of $f(x)$ and $f(y)$, where $y = f(x)$. And since f represents the same quadratic function, Matei is correct in drawing two parabolas (in fact, the same parabola) to first represent f as a function of x, and then to represent f as a function of $f(x)$.

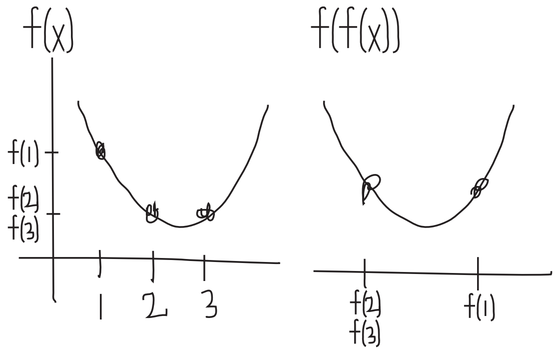
4. In your own words, interpret the drawing in line 21. Then make similar diagrams for when $f(1) = f(3)$ and for when $f(2) = f(3)$.

   The first drawing on line 21 is of the function $f(x)$ with values of x on the horizontal axis. In this drawing, we see that the inputs 1 and 2 produce the same output $f(1) = f(2)$. The second drawing on line 21 is of $f(f(x))$ with values of $f(x)$ on the horizontal axis. The two graphs are of the same parabola. This time, the values of the inputs that are labeled are the same as the values of the outputs from the first graph. We see that $f(1)$ and $f(2)$ are the same number and their output, $f(f(1)) = f(f(2))$ is the same as the output when the input is $f(3)$.

Here is a corresponding drawing for when $f(1) = f(3)$:

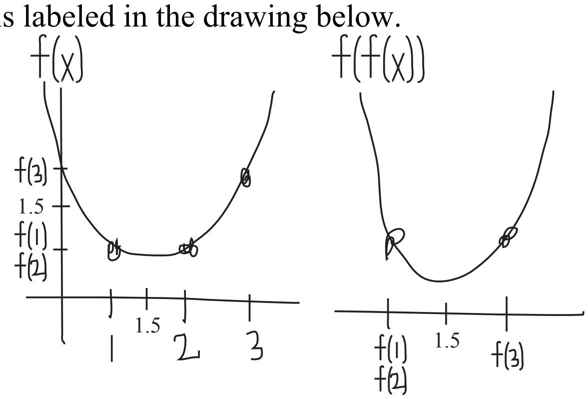


And a drawing for when $f(2) = f(3)$:

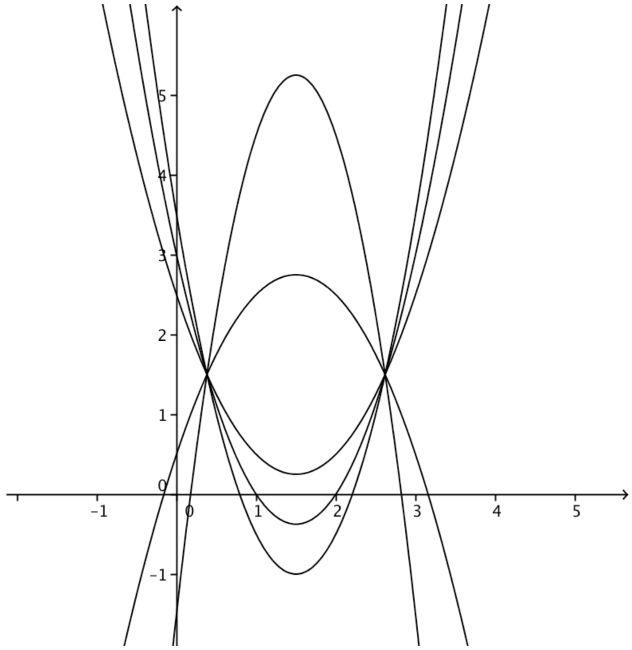


5. The second part of the task is to determine *all* of the quadratic functions that satisfy the relationship $f(f(1)) = f(f(2)) = f(f(3))$. Predict the number of solutions there are. Does *any* quadratic function work?

   There are three different cases to consider: when $f(1) = f(2)$, $f(2) = f(3)$, and $f(1) = f(3)$. In each case, a family of functions will satisfy the relationship. For example, consider the case when $f(1) = f(2)$, shown by Matei's drawing in line 21. Because of the symmetry of a parabola, we can fill in some more details to help us visualize the possible parabolas that would satisfy the relationship. Because $f(1) = f(2)$, we know that the vertex of the graph of $f(x)$ occurs at $(1.5, f(1.5))$. That also means that 1.5 is the average of $f(1) = f(2)$ and $f(3)$. The value 1.5 is labeled in the drawing below.



   From this picture, it is possible to see that a family of parabolas can be made to fit these constraints, all with a vertex at $(1.5, f(1.5))$. A few of these parabolas are shown below.



A similar family of quadratic functions can be constructed for each of the two other cases where $f(2) = f(3)$ and $f(1) = f(3)$. So we expect the solution to be three families of quadratic functions. There are infinitely many functions in each family.

6. Finishing this task requires a lot of algebraic manipulation. If you are already well on your way towards a solution, keep going! If you would like a little guidance along the way, the following questions may help.
   (a) Find $f(1)$, $f(2)$, and $f(3)$ in terms of a, b, and c.
   (b) If $f(1) = f(2)$, then that constrains the relationship between a and b. Use that constraint to rewrite $f(1)$, $f(2)$, and $f(3)$.
   (c) Find $f(f(1))$, $f(f(2))$, and $f(f(3))$ in terms of a and c. (Remember, you can write b in terms of a.) Finally, figure out how a, b, and c must be related for $f(f(1)) = f(f(2)) = f(f(3))$.
   (d) Repeat the problem for the case where $f(1) = f(3)$.
   (e) And again for the case where $f(2) = f(3)$.
   (f) State your solution to the task in a way that makes it clear you know what you've found as a result of all of that algebra.

See the Mathematical Overview for the solution.

Possible Responses to Related Mathematics Tasks

1. For a quadratic function of the form $f(x) = ax^2 + bx + c$, determine whether or not it's possible that $f(1) = f(3)$ and $f(2) = f(4)$.

   No. Students can use symmetry to show that this is not possible. For instance, if $f(1) = f(3)$, then the vertex of the parabola must be at the point $(2, f(2))$, and so it is not possible that $f(2) = f(4)$.

2. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$, and let $g(x)$ be a linear function of the form $g(x) = px + q$. Is it possible that $f(g(1)) = f(g(2)) = f(g(3))$? Explain.

   It is possible that $f(g(1)) = f(g(2)) = f(g(3))$ if $g(1) = g(2)$ *or* $g(1) = g(3)$ or $g(2) = g(3)$. Since $g(x)$ is a linear function, two different inputs will produce the same output only if the function is constant. In other words, $g(x) = q$.

3. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$, and let $g(x)$ be a linear function of the form $g(x) = px + q$. Is it possible that $g(f(1)) = g(f(2)) = g(f(3))$? Explain.

   If $g(x)$ is a constant function (i.e., $g(x) = q$), then $f(x)$ can be *any* quadratic function and this relationship will hold. If $g(x)$ is not a constant function, then the only way the relationship can be satisfied is if $f(1) = f(2) = f(3)$. But we found in this Illustration that this is not possible for a quadratic function. So $g(f(1)) = g(f(2)) = g(f(3))$ holds only if $g(x)$ is a constant function.

4. Let $f(x)$ be a quadratic function of the form $f(x) = ax^2 + bx + c$. Is it possible to find p, q, r, and s such that $f(f(p)) = f(f(q)) = f(f(r)) = f(f(s))$? Explain.

   Yes. Assuming that $p < q < r < s$, it would have to be true that $f(q) = f(r)$ and $f(p) = f(s)$, and then that $f(f(p)) = f(f(q))$.

5. In this Illustration, students make sense of $f(f(x))$ by seeing that they can think of the structure as a "problem-within-a-problem." This kind of reasoning shows up in mathematics surprisingly often. Here is another task that you might approach by taking advantage of its problem-within-a-problem structure:

$$\text{Find } \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$$

   Let $x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$. Then $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = \sqrt{2+x}$. So $x = \sqrt{2+x}$. We can solve this equation by squaring both sides ($x^2 = 2 + x$), then solving the quadratic equation ($x^2 - x - 2 = 0$) by factoring ($(x - 2)(x + 1) = 0$). There are two solutions $x = 2$ and $x = -1$. But the original quantity $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$ must be positive. So the original quantity equals 2.