Multiplying Two Fractions

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Multiplying Two Fractions* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to make sense of why they get the answer they get when multiplying two fractions using the common algorithm they have learned, and in particular why they multiply both the numerators and the denominators. They reason about the quantities that each fraction refers to (e.g., 2/3 is two one-thirds) and also use an area model to visually represent multiplication when their previous approach becomes problematic.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grade 5

Target Content Domain: Number and Operations—Fractions

Highlighted Standard(s) for Mathematical Content

5.NF.B.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

5.NF.B.4a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)*

5.NF.B.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

Math Topic Keywords: fractions, multiplication, unit fractions

This material is based on work supported by the National Science Foundation under Grant No. DRL-1119163. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

What is $\frac{6}{5} \times \frac{2}{3}$?

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students already know a standard algorithm for multiplying two fractions and have experience using area models for fraction multiplication as well. They know how to rewrite a fraction as a unit fraction times an integer and understand what that means. That is, they easily interpret a fraction like $\frac{3}{5}$ *as meaning "three* ***fifths****," which is to say three of the fraction* $\frac{1}{5}$*. They use the same understanding to know, without rewriting it and applying an algorithm, that* $4 \times \frac{3}{5} = \frac{12}{5}$ *because "4 times 3* ***anything*** *would be 12* ***anything****, so 4 times three* ***fifths*** *would be 12* ***fifths****." For them, this bit of raw arithmetic could be a routine exercise, answered instantly, but they begin to wonder about it, raising questions on their own. They start their Student Dialogue using one line of reasoning, and later switch strategically to a different line of reasoning when they encounter a situation in which their first approach feels cumbersome.*

(1) Dana: Well this is easy. All we need to do is multiply the top and bottom numbers: 6 times 2 is 12 and 5 times 3 is 15 so our answer is $\frac{12}{15}$.

(2) Anita: You could also write $\frac{12}{15}$ as $\frac{4}{5}$. That's just another name for the same number.

(3) Sam: I know this works, but I still don't understood *why*. And why do we multiply both the top *and* the bottom numbers? When we add fractions we add only the top numbers and we do nothing to the bottom number.

(4) Dana: Well, no. That thing you said about adding is true only if the two denominators are the same. If the denominators are different, it's more complicated.

(5) Sam: OK, right, but that's the point. Each thing we do seems to have its own special rule. We figured out why the addition rule works. I want to know why *this* rule works.

(6) Anita: Well, $\frac{2}{3}$ is 2 thirds. Like two cats. We have two of them.

(7) Sam: Cats. Oh, great. So $\frac{6}{5}$ is 6 of those fifths. Why not goats? How does this help?!

(8) Anita: Well, let's think of *six* fifths multiplied by 2, that gives us *twelve* fifths.

(9) Sam: Yes, but we aren't multiplying by 2. We're multiplying by $\frac{2}{3}$.

(10) Anita: Well, $\frac{2}{3}$ is just a third of 2. So you should get a third of what you get when you multiply by two.

(11) Sam: Wait. First you're calling $\frac{2}{3}$ two thirds, like two cats. Now you're calling it a third of 2. You're really losing me.

(12) Anita: Sorry. You're right. It's both, but I wasn't really being clear. Here, when I said $\frac{2}{3}$ is *two* thirds, I was thinking 2 times *one* third. But when I write out $2 \times \frac{1}{3}$, I can also see it as meaning one third of 2. Now, what was I saying before?

(13) Dana: I've got it! You were saying that *six* fifths times 2 is *twelve* fifths, so…

(14) Sam: Ah, I've got it, too! *Six* fifths times 2 is *twelve* fifths, but we're not multiplying by *two*, we're multiplying by *one third of 2*, so…we first multiply by 2, then we multiply by $\frac{1}{3}$.

(15) Dana: So if we get *twelve* fifths when we multiply by the number *two,* we would get *one third* of that when we multiply by the number $\frac{2}{3}$. OK, so we need to take one third of those $\frac{12}{5}$, which would be *four* fifths. So two thirds of six fifths is $\frac{4}{5}$.

(16) Sam: That's the same as what you guys got when you multiplied the tops and the bottoms and then found the equivalent fraction with the smallest denominator.

(17) Anita: Yes! And that gave me an idea about another way to think about this.

(18) Sam: Stop!

(19) Anita: *Two* thirds of 6 fifths is twice as much as *one* third of 6 fifths.

(20) Dana: And one third of *six* fifths is *two* fifths! Cool! So *two* thirds of 6 fifths is twice as much, *four* fifths.

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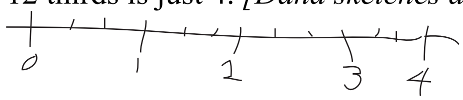
(21) Sam: I see. What if we try this with the first fraction. *Six* times 2 thirds is 12 thirds, so six *fifths* times 2 thirds would be a fifth of that…. Ewww. What's a fifth of 12 thirds?

(22) Dana: 12 thirds is just 4. *[Dana sketches a rough number line.]*



(23) Sam: Oh, right! And a fifth of 4 is $\frac{4}{5}$. That's what we got before.

(24) Anita: It better be!

(25) Sam: And what if we try it this way? $\frac{6}{5}$ is just 6 times *one* fifth, so let me start with *one* fifth of 2 thirds…. What is that going to be? Wait! I don't know how to think about this one. We can't easily do what we did before.

(26) Dana: Yuck! That *is* ugly.

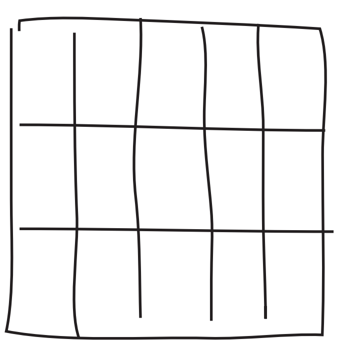
(27) Anita: Even though we don't know what that's going to be, we know that once we multiply it by 6 it should equal 4 fifths.

(28) Sam: That makes sense but that doesn't really help us here, does it?

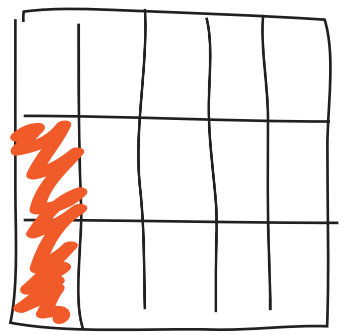
(29) Anita: I guess not.

(30) Dana: So let's try using an area model for this situation. We're multiplying $\frac{1}{5} \times \frac{2}{3}$, so…

(31) Sam: OK. *[Talks to himself as he draws.]* So we start with a 1 by 1 square. *[draws square]* And then mark the bottom as fifths and draw… *[draws five roughly equal columns]*… and now I mark the side as thirds… *[draws the three roughly equal rows]* OK, done!



(32) Dana: Yup, it's a 1 by 1 square, so its area is 1. And that area is cut into 15 pieces, so each tiny piece is $\frac{1}{15}$. Now we shade the rectangle that measures $\frac{1}{5} \times \frac{2}{3}$. *[shades in the rectangle]*



The rectangle has 2 pieces in it, and so its area is $\frac{2}{15}$.

(33) Sam: All right. So now we know that *one* fifth times $\frac{2}{3}$ is $\frac{2}{15}$.

(34) Anita: Yes. And to figure out *six* fifths times $\frac{2}{3}$ we need to multiply by 6, which gives us 12 fifteenths.

(35) Sam: That's a completely different answer. Oh! But we can reduce $\frac{12}{15}$ to… $\frac{4}{5}$. This works too!

(36) Anita: As it should!

(37) Sam: Actually, couldn't we have just used the area model for the original problem, too?

(38) Anita: Sure. And this gave me an idea for another way to figure out how to *add* fractions with unlike denominators. Here, let me show you!

(39) Sam and Dana: Lunch first, then new ideas!

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. In line 3, Sam says that to add fractions one adds only the numerators. Dana clarifies that this works only with like denominators, and otherwise "it's more complicated." What approach seen in the Student Dialogue can help explain why adding the numerators makes sense as a way to add like denominator fractions and doesn't make sense with unlike denominators?

3. In the Student Dialogue, students think about fractions as numerators, counting the number of unit fractions indicated by the denominator. How can this mode of counting (1) build off of students' prior knowledge about the number system and (2) set the stage for future learning about the number system?

4. At the end of the Student Dialogue, the students mention they could have used an area model for the original problem. Solve $\frac{6}{5}\times\frac{2}{3}$ using an area model. What new insights about area models might students encounter in using an area model for $\frac{6}{5}\times\frac{2}{3}$?

5. Students discuss three different methods for multiplying fractions: (1) the standard algorithm, (2) reasoning about the quantity each fraction refers to, and (3) using an area model. List some advantages and disadvantages of each method.

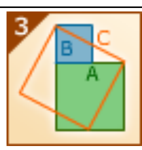
6. In this Student Dialogue, fractions are treated as quantities with units (e.g., $\frac{6}{5}$ is 6 fifths), and it appears as if units are being multiplied to create new units (e.g., 6 fifths × 2 thirds = 12 fifteenths). Is it possible to treat multiplication with other units in the same way (for example, 3 inches × 5 feet = 15 inch•feet)? What restrictions, if any, exist on being able to multiply two units in this way?

7. For each of the following units, try to invent some sensible interpretation:
   A) 1 man•hour
   B) 1 passenger•mile
   C) 1 hen•egg
   D) 1 Newton•meter
   E) 1 cat•dog

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Make sense of problems and persevere in solving them.	Sam's question in line 3 is about making sense of the mathematics. Sam is not satisfied simply with getting the answer; Sam wants to understand. Elements of MP 1 such as "check their answers to problems using a different method" can also be seen in lines 23 and 24, where students check to make sure that the solution they come up with using a different approach matches their previous findings. For the students, it makes sense that there is a unique solution to the problem, and they even use that to "conjecture about the form … of the solution" to related problems. For example, in line 27 Anita conjectures that even though they don't know what 1 fifth of 2 thirds is, they know that if you "multiply it by 6 it should equal 4 fifths" since they have already derived that result with another approach. Furthermore, students use multiple meanings of fractions (e.g., in line 12, seeing $\frac{2}{3}$ as 2 *thirds* or as one third of 2) in their pursuit of "try[ing]… simpler forms of the original problem."
 Construct viable arguments and critique the reasoning of others.	Throughout, we see students building arguments. They have a solution (lines 1–2) from an algorithm that is not understood, and the students "build a logical progression of statements to explore the truth of their conjectures" (i.e., first solution). Examples of this logical progression of statements can be found on lines 7–15 and 19–23, where students explain through various arguments (that they do understand!) why $\frac{6}{5}\times\frac{2}{3}=\frac{4}{5}$.
 Use appropriate tools strategically.	Students begin with a strategy based on reasoning about units. When that becomes awkward ($\frac{1}{5}\times\frac{2}{3}$ in line 30), they change their solution strategy (MP 1) and turn strategically to a different tool—area models—to help them solve the problem. Students show that they can "make sound decisions about when… tools might be helpful."

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Look for and make use of structure.	Students use the structure inherent in fractions as they reason through their different arguments. They interpret the relationship between the numerator and denominator in different ways. One interpretation is that fractions are a division of integers as seen in line 10 where $\frac{2}{3}$ is interpreted as a third of 2 or in line 22 where Dana realizes 12 thirds is simply 4. Another interpretation is that fractions are built from unit fractions as seen in line 6 where $\frac{2}{3}$ is said to be "2 thirds. Like two cats. We have two of them." This particular view of fractions can also be seen as an instance of MP 2 since students are "considering the units involved" and "attending to the meaning of quantities." The use of multiple interpretations of $\frac{2}{3}$ (i.e., 2 thirds or a third of 2) can also be attributed to "flexibly using different properties of operations" (MP 2), specifically the commutativity of multiplication in $2\times\frac{1}{3}$ (line 12).

Commentary on the Mathematics

Though MP 1, MP 3, MP 5, and MP 7 are the most prominent in the students' way of thinking in this Student Dialogue, they make fleeting use of MP 2, MP 6, and MP 8, as well. In fact, the only aspect of mathematical practice that is *not* particularly evident in the thinking of the students in this Student Dialogue is MP 4: using mathematics to model a phenomenon. In the Student Dialogue, students use an area diagram as a model of fraction multiplication, and while that *is* a case of using the mathematics of geometry to model the mathematics of number, it is not the intent of "model with mathematics" in MP 4. It is also worth noting that while most Standards for Mathematical Practice may be found in this particular Student Dialogue, to varying degrees, it does not mean all eight standards will be present in most cases. We should not overgeneralize and fall in the trap of seeing all the Standards for Mathematical Practice all the time.

Students' thinking moves back and forth between pure numbers (the fractions) and analogies to counting *things* (cats). This is not a "deep" use of thinking abstractly and quantitatively (MP 2), but it *is* an example of that kind of thinking. Students move fluidly between interpreting numbers like $\frac{2}{3}$ as an abstract number, itself, and as a quantity of something else, a repetition of the units we call *thirds*, in much the same way we would talk about two inches or two hours.
Anita in line 6 focuses on the precision of a statement in their discussion, and later in line 12 also refines her statement (MP 6) in response to Sam's implicit critique (line 11) of the logical completeness and coherence of her argument (MP 3). For most of the middle third of this Student Dialogue, the students are repeating similar calculations (MP 8) not so much to *express* the regularity in some abstract way (e.g., with algebra) but—by taking these calculations apart and repeating parts with simpler numbers—to *find* that regularity and use it as a structure to further their reasoning.

Multiplying Two Fractions

Perhaps the most constant mathematical thread in this Student Dialogue is the students' flexibility in thinking about fractions and their perseverance in using multiple interpretations of fractions in their determination to make sense of the computation they were performing (MP 1). They were all satisfied at the start that the algorithm worked, but when Sam raised the question about *why* it worked, they all dug in. This is a perfect example of students who already know how to do a piece of routine arithmetic deciding to tackle a piece of mathematics instead, *deriving* a method rather than just using it. When this happens, it is good evidence that students' history, in at least *this* class, taught them that it is both permissible and smart to re-examine ideas that they feel like they already fully understand. They don't *have* to question their own understanding—it would be impractical, not to mention distracting, even devastating, to do that constantly—but they *may*, and that is what happens here when they try to make new sense out of old ideas by combining the old ideas in new ways.

Sometimes (e.g., in lines 6, 7, and 12) they interpret a fraction as an enumeration (how many) of a fixed denomination unit, a kind of measurement of a quantity. That is, they treat a fraction as an integer times a unit fraction: $\frac{2}{3}$ is 2 thirds or $2\times\frac{1}{3}$, and $\frac{6}{5}$ is 6 fifths or $6\times\frac{1}{5}$. At other times (e.g., in 10, 12, 14, and 15), they see a fraction as specifying a division of one integer by another. These two ways of "taking a fraction apart" give more sense to the fractions themselves and are aligned to the multiple modes of thinking around fractions as required by the *Common Core* (5.NF.B.3 and 4.NF.B.4a).

They also—and it is their initial goal—attempt to make sense of the computation. Both in interpreting the fractions themselves and interpreting the computation, they use the *ideas* of ratio or proportion, though without ever using those *words*. For example, they make sense of the computation $\frac{2}{3}\times\frac{6}{5}$ by treating it as scaling the computation $\frac{6}{5}\times 2$ by one third. (Note that students are also using the commutative property and view $\frac{6}{5}\times\frac{2}{3}=\frac{2}{3}\times\frac{6}{5}$.) By viewing fractions as an integer times a unit fraction and using the commutative property, students can think of multiplying by a fraction in two ways. The first is to multiply by the whole number (numerator) and then the unit fraction (as in line 14). The second approach is to multiply by the unit fraction followed by the whole number (as in lines 19 and 20). In fact, breaking up fractions into products of whole numbers and unit fractions and rearranging them using the commutative product can simplify calculations. In the case of $\frac{4}{5}\times\frac{5}{8}$, we can rewrite the product as $4\times\frac{1}{5}\times 5\times\frac{1}{8}$ and use the commutative property to rearrange the factors to $4\times\frac{1}{8}\times 5\frac{1}{5}=\frac{4}{8}\times\frac{5}{5}=\frac{1}{2}$.

As one of the several ways of sense-making that these students are using, Sam also constantly checks to make sure the results they are getting are consistent (lines 16, 23, 35). And they give meaning priority over form. When Anita (line 12) equates "*two* thirds [meaning]… 2 times *one* third" with "one third of 2," she explains it just as two interpretations of the expression $2\times\frac{1}{3}$, without bothering to rewrite the expression as $\frac{1}{3} \times 2$ (using the commutative property) to make the written order conform to the words she speaks; she knows that the order in which it is *written* doesn't restrict the way in which it can be interpreted.

Evidence of the Content Standards

In the Student Dialogue, students attempt to understand why the standard algorithm for fraction multiplication works based on what they already know about fractions and multiplication (5.NF.B.4). Students also flexibly interpret fractions. For example in line 12, Anita explains how $\frac{2}{3}$ is equivalent to $2 \times \frac{1}{3}$ (5.NF.B.4a). And Dana in line 22 is able to identify 12 thirds ($\frac{12}{3}$) as 4 by dividing 12 by 3 (5.NF.B.3). Area models are also used to reason about the multiplication of two fractions (5.NF.B.4a).

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. Before students in the dialogue begin calculating the product of the two fractions, how might they estimate the product?

2. In lines 3 and 4, Sam and Dana discuss what to do when adding two fractions. Why is it that when adding like denominator fractions, you can add the two numerators and keep the same denominator?

3. In lines 6 through 12, the students in the Student Dialogue explain two different ways to think about the fraction $\frac{2}{3}$. Using your own words, explain at least two ways to think about the fraction $\frac{8}{11}$.

4. In line 24, Anita says that Sam's answer "better be" the same as her own, even though Sam used a different method. What if two methods give *different* answers? What can you say about at least one of the methods?

Related Mathematics Tasks

1. At the end of the Student Dialogue, the students mention they could have used an area model for the original problem. Solve $\frac{6}{5}\times\frac{2}{3}$ using an area model. What new insights about area models might students encounter in using an area model for $\frac{6}{5}\times\frac{2}{3}$?

2. Solve $\frac{3}{2}\times\frac{4}{9}$ mentally in at least two different ways by reasoning about quantities, similar to the way students did in the Student Dialogue. Once you have solved the problem mentally, explain each of your ways in words and write an expression or equation that summarizes each process.

3. Solve $\frac{4}{7} \times \frac{2}{3}$ mentally by reasoning about quantities, similar to the way students did in the Student Dialogue. What challenges arise in this particular example? Explain.

4. Use some method to show *why* the standard algorithm calculation of $\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}$ produces the correct answer.

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. In line 3, Sam says that to add fractions one adds only the numerators. Dana clarifies that this works only with like denominators, and otherwise "it's more complicated." What approach seen in the Student Dialogue can help explain why adding the numerators makes sense as a way to add like denominator fractions and doesn't make sense with unlike denominators?

   When Sam makes a statement about adding fractions that is only true under certain conditions, Dana jumps in adding precision to Sam's overgeneralized rule. To help make sense of the meaning of fraction, Anita, in line 6, compares two thirds to two cats. (The numerator *enumerates*.) So, just as 4 cats plus 3 cats equals 7 cats and 4 hours plus 3 hours are 7 hours, 4 fifths plus 3 fifths equals 7 fifths. But if the "denominations" are different—for example, hours and inches, or flavors and umbrellas—just adding the *numbers* of them makes no sense. You can't add 4 of one and 3 of the other to get 7 of something. We can proceed with the addition only if we can find a commonality among the units—for cats and dogs it might be animals, and for thirds and fifths it might be fifteenths. For more about fractions with unlike denominators, see the Illustration *Anita's Way to Add Fractions with Unlike Denominators*.

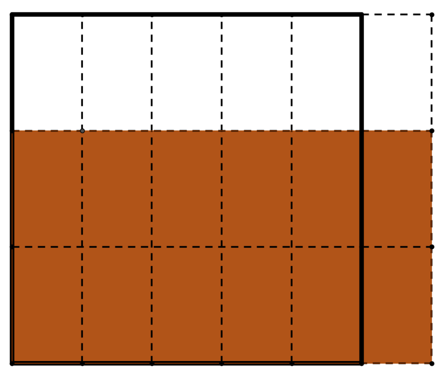
3. In the Student Dialogue, students think about fractions as numerators, counting the number of unit fractions indicated by the denominator. How can this mode of counting (1) build off of students' prior knowledge about the number system and (2) set the stage for future learning about the number system?

   Counting by unit fractions can be connected to students' prior experience with skip-counting (2.NBT.A.2) and with place value where the digit in a number represents how many ones, tens, hundreds, etc., fit into that number. Also, counting based on some reference (e.g., a multiple, unit fractions) comes up when students are learning about decimals and their place value (5.NBT.3).

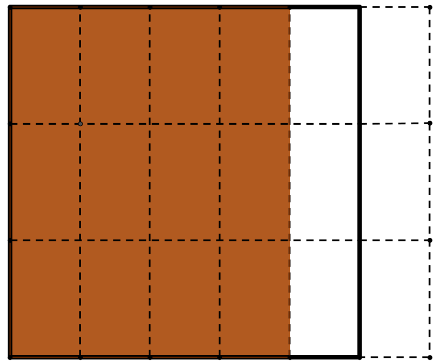
   In this Student Dialogue, the students connect their understanding of counting the number of unit fractions as counting other things in everyday life such as cats and goats (lines 6 and 7). This comparison may be made by some students and have the benefit of not distracting them with a reference object that is also a number. They are focused on the fact that they "have two of them" (line 6) without having to worry about the fact that they are counting the number of thirds or that $\frac{2}{3}$ is also a number itself.

4. At the end of the Student Dialogue, the students mention they could have used an area model for the original problem. Solve $\frac{6}{5}\times\frac{2}{3}$ using an area model. What new insights about area models might students encounter in using an area model for $\frac{6}{5}\times\frac{2}{3}$?

The area model of multiplication—for integers or for rational numbers—compares a rectangle's area to the area of a 1 by 1 square. To represent $\frac{6}{5}\times\frac{2}{3}$ with an area model, one needs a consistent sketch of a rectangle that measures $\frac{6}{5}$ by $\frac{2}{3}$. In the sketch below, each small dotted rectangle represents a fifteenth (because 15 of them fit in the 1×1 square); the shaded $\frac{6}{5}$ by $\frac{2}{3}$ rectangle contains 12 of those fifteenths, so $\frac{6}{5}\times\frac{2}{3}=\frac{12}{15}$.



The $\frac{6}{5}$ must, of course, extend beyond the sides of the 1×1 square and this might be a new insight for students who have experience using area models only for fractions smaller than 1. Students must break the unit square into fifths in one direction and then add an *extra* fifth to represent that 6^{th} fifth. Once 12 dotted rectangles are shaded, students must interpret what the 12 rectangles mean. Some might mistakenly conclude that the model shows $\frac{12}{18}$ since there are 18 dotted rectangles. This is a "useful error." The value of the dotted rectangle is not based on how many there are, but how many of them are *within a unit square* (1). Because 15 of the small shaded rectangles fill that unit square, each one of them represents a fifteenth. The picture shows 12 of them shaded, giving us a solution of $\frac{12}{15}$. Another approach students may use after building their area model is to move the 4 rightmost dotted shaded rectangles to the top row, resulting in the image below.



With this rearranged area model, students can see that $\frac{4}{5}$ of the unit square is shaded.

While this approach avoids the issue of choosing between 15 or 18 as the denominator of the fraction, students still need to understand that the fraction of the shaded area is determined with respect to the unit square (i.e., $\frac{4}{5}$ *of the square,* not $\frac{4}{6}$ *of the entire diagram*).

Note that this problem is posed to students in the Related Mathematics Tasks.

5. Students discuss three different methods for multiplying fractions: (1) the standard algorithm, (2) reasoning about the quantity each fraction refers to, and (3) using an area model. List some advantages and disadvantages of each method.

   This question is also useful to use with students (see Student Discussion Questions).

   The standard algorithm is a fast, easy shortcut that can be used with any two fractions. Shortcuts that are memorized without understanding, though, are easy to confuse with each other (see Sam's question, line 3) leading to errors.

   Reasoning about quantities (two thirds, like two cats) interprets the calculations in terms of integer multiples of unit fractions—one important way of understanding fractions. However, reasoning about quantities requires that one find common factors (see $\frac{1}{5}\times\frac{2}{3}$ in line 30).

   Using an area model visually organizes and shows commonality among all multiplications including multiplication of fractions, whole numbers, and algebraic expressions, and helps make the processes understandable. Its only possible disadvantage is the time it takes to set up, so we generally switch to other methods once we understand them. However, even when other methods are understood and preferred, we may still use a sketch of an area model to help us organize our thinking about what we need to multiply and count.

6. In this Student Dialogue, fractions are treated as quantities with units (e.g., $\frac{6}{5}$ is 6 fifths), and it appears as if units are being multiplied to create new units (e.g., 6 fifths × 2 thirds = 12 fifteenths). Is it possible to treat multiplication with other units in the same way (for example, 3 inches × 5 feet = 15 inch•feet)? What restrictions, if any, exist on being able to multiply two units in this way?

   Though numbers of units cannot be *added* if the units, themselves, differ, it is possible to *multiply* two units that differ, as long as the resulting unit can be given a meaning that is useful. For example, we can multiply 3 inches by 5 feet to get 15 inch•feet, and we know that equates to some area. However, making sense of the unit inch•foot is tricky since it differs from the common units of square inches or square feet. An inch•foot can be interpreted as the area of a rectangle that is 1 inch wide and 1 foot long. In short, new units can often be derived by multiplying other units; the challenge lies in giving the derived units meaning.

7. For each of the following units, try to invent some sensible interpretation:

   A) 1 man•hour
   B) 1 passenger•mile
   C) 1 hen•egg
   D) 1 Newton•meter
   E) 1 cat•dog

      A) Man•hour is a commonly used unit, meaning the number of hours of work done. One man•hour represents the amount of work 1 person can do in 1 hour. Five people working for 2 hours each, or 2 people working for 5 hours each, or 1 working for 10 hours, all would be called "10 man•hours."
      B) Passenger•mile could describe the total distance traveled by a group of people. One passenger•mile represents 1 mile traveled by 1 passenger. The more passengers in a vehicle or the farther the destination, the more passenger•miles.
      C) We might invent the unit hen•egg to describe the total number of eggs produced. One hen•egg might equate to 1 hen making only 1 egg. The more hens one has or the more eggs each hen produces, the more hen•eggs you have. The unit seems pretty useless, though! The total number of eggs is the only information we need.
      D) Newton•meter represents the total force used to move an object. One Newton•meter means it took 1 Newton of force to move an object 1 meter. The more Newtons it takes to move an object a distance of 1 meter or the more meters the object is pushed, the greater the Newton•meters. In fact the Newton•meter is abbreviated to Joule in physics and is used when measuring work (in the physics sense).
      E) How can we give a meaning to the made-up unit cat•dog? It could represent the number of cat•dog pairings. Such a unit may be used in the scenario of someone

entering a pet store knowing they will leave with a cat and a dog. If there are 3 cats and 2 dogs to choose among, there are six possible pairs that could be made. The number of possibilities could be given the unit cat•dog.

Of course, this exercise does not advocate hen•egg or cat•dog units. The purpose is only to show how it's possible to give meaning to products of units.

Possible Responses to Student Discussion Questions

1. Before students in the dialogue begin calculating the product of the two fractions, how might they estimate the product?

   Looking at the two factors, they can see if the fractions are greater than, less than, or equal to 1. If both are greater, then you could estimate the product if greater than 1, too, and if both are smaller, you could estimate that the product is less than 1, too. However, in this case, you have one fraction that is greater than 1, $\frac{6}{5}$, and one fraction less than 1, $\frac{2}{3}$. Since $\frac{2}{3}$ is farther from 1 than $\frac{6}{5}$ is from 1, that means the product will also be smaller than 1.

2. In lines 3 and 4, Sam and Dana discuss what to do when adding two fractions. Why is it that when adding like denominator fractions, you can add the two numerators and keep the same denominator?

   You can think of fractions as quantities with units of thirds, fifths, etc. When two fractions have the same denominator, they are quantities with the same units that can simply be added together. 4 fifths and 3 fifths = 7 fifths, same as 4 cats and 3 cats = 7 cats.

3. In lines 6 through 12, the students in the Student Dialogue explain two different ways to think about the fraction $\frac{2}{3}$. Using your own words, explain at least two ways to think about the fraction $\frac{8}{11}$.

   We can think of $\frac{8}{11}$ as 8 elevenths, which we might write as $8\times\frac{1}{11}$. Or we can think of $\frac{8}{11}$ as $\frac{1}{11}$ of eight, which we might write as $\frac{1}{11}\times 8$. And, of course, $8\times\frac{1}{11} = \frac{1}{11}\times 8$ for the same reason that 2 × 7 = 7 × 2 or any other multiplication can be switched around. Some students may know the term "commutative property." Other students may think of equivalent fractions and equate $\frac{8}{11}$ to $\frac{16}{22}$ or some other equivalent fraction.

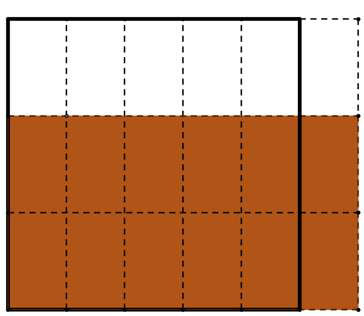
4. In line 24, Anita says that Sam's answer "better be" the same as her own, even though Sam used a different method. What if two methods give *different* answers? What can you say about at least one of the methods?

   Anita expects that $\frac{6}{5}\times\frac{2}{3}$ has only one correct answer even if that answer may be written in different ways as equivalent fractions or perhaps in a decimal representation. If two methods give genuinely different answers (not just two forms of the same number), at least one of them must certainly be wrong. They might *both* be wrong, of course. Getting the same answer, even the *right* answer, with two different methods does not guarantee that the methods are *right*! A very famous example is "cancelling the 6" in $\frac{16}{64}$ to get $\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4}$. The answer is correct, $\frac{16}{64}=\frac{1}{4}$, but the method won't work with most fractions. For example, $\frac{26}{64}\neq\frac{2}{4}$. The method is just wrong, even though it "works" in this one special case. (Can you find any other cases for which it "works"?)

Possible Responses to Related Mathematics Tasks

1. At the end of the Student Dialogue, the students mention they could have used an area model for the original problem. Solve $\frac{6}{5}\times\frac{2}{3}$ using an area model. What new insights about area models might students encounter in using an area model for $\frac{6}{5}\times\frac{2}{3}$?

   The area model of multiplication—for integers or for rational numbers—compares a rectangle's area to the area of a 1 by 1 square. To represent $\frac{6}{5}\times\frac{2}{3}$ with an area model, one needs a consistent sketch of a rectangle that measures $\frac{6}{5}$ by $\frac{2}{3}$. In the sketch below, each small dotted rectangle represents a fifteenth (because 15 of them fit in the 1 by 1 square); the shaded $\frac{6}{5}$ by $\frac{2}{3}$ rectangle contains 12 of those fifteenths, so $\frac{6}{5}\times\frac{2}{3}=\frac{12}{15}$.



2. Solve $\frac{3}{2}\times\frac{4}{9}$ mentally in at least two different ways by reasoning about quantities, similar to the way students did in the Student Dialogue. Once you have solved the problem mentally, explain each of your ways in words and write an expression or equation that summarizes each process.

   One way: *one* half of 4 ninths is 2 ninths, but the problem asks for *three* halves, so we triple the result, multiplying the 2 ninths by 3. This gives us 6 ninths. We can write these steps as two equations: $\frac{1}{2}\times\frac{4}{9}=\frac{2}{9}$ and $3\times\frac{2}{9}=\frac{6}{9}$.

   Another way: The $\frac{3}{2}$ is equal to $1\frac{1}{2}$. The computation then asks for $1\frac{1}{2}$ of 4 ninths. That is 4 ninths plus half of 4 ninths, a total of 6 ninths. These steps can be expressed as $\frac{3}{2}=1\frac{1}{2}$ and $1\frac{1}{2}\times\frac{4}{9}=\left(1\times\frac{4}{9}\right)+\left(\frac{1}{2}\times\frac{4}{9}\right)=\frac{4}{9}+\frac{2}{9}=\frac{6}{9}$.
   Because of the numbers, this problem is also fairly easy to do mentally just by noticing the factors one can "cancel," simplifying $\frac{3}{2}\times\frac{4}{9}$ to $\frac{1}{1}\times\frac{2}{3}$.

3. Solve $\frac{4}{7}\times\frac{2}{3}$ mentally by reasoning about quantities, similar to the way students did in the Student Dialogue. What challenges arise in this particular example? Explain.

   This problem is difficult to solve, reasoning about quantities since the numbers do not have common factors. For example, 4 times 2 thirds gives us 8 thirds, but what is a seventh of 8 thirds? One can certainly think "what is a seventh of a third" and reach $\frac{1}{21}$ that way, so a seventh of *eight* thirds would be eight times that amount. Or we can take 4 sevenths and multiply by 2 to get 8 sevenths, but then what is a third of that? Similar reasoning will again give us $\frac{8}{21}$, but this example is certainly more challenging than $\frac{3}{2}\times\frac{4}{9}$.

4. Use some method to show *why* the standard algorithm calculation of $\frac{4}{7}\times\frac{2}{3}=\frac{8}{21}$ produces the correct answer.

   We can use an area model. Break a unit square into sevenths along one side and thirds along the other, shading in a region that measures 4 sevenths by 2 thirds. This will also give us a solution of $\frac{8}{21}$, and also shows where the numerator and denominator come from. The product of the numerators, 4×2, describes the shaded region; the product of the denominators, 7×3, describes the whole region.