Proof with Parallelogram Vertices

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Proof with Parallelogram Vertices* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to prove a conjecture they've made regarding the location of the possible fourth vertices of a parallelogram given only three vertices. As students try to prove their conjecture they struggle with proving the collinearity of points and begin intuitively using the parallel postulate.

Highlighted Standard(s) for Mathematical Practice (MP)

MP 1: Make sense of problems and persevere in solving them.
MP 3: Construct viable arguments and critique reasoning of others.
MP 6: Attend to precision.

Target Grade Level: Grades 9–10

Target Content Domain: Congruence (Geometry Conceptual Category)

Highlighted Standard(s) for Mathematical Content

G-CO.C.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

G-CO.C.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Math Topic Keywords: proof, parallelograms, midpoint, 180º in a line, transversals, 180º in a triangle

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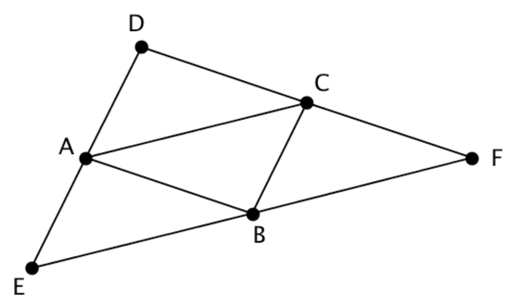
Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Given three non-collinear points, A, B and C, as vertices, students found that they could form a parallelogram by placing the fourth vertex at any one of exactly three positions, points D, E and F. Students noticed that A, B and C seem to be midpoints of the sides of the newly formed $\triangle DEF$. How would you prove their conjecture?



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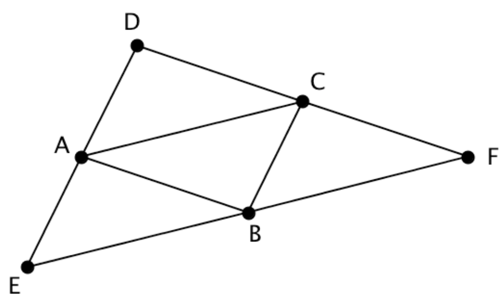
Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have been studying proofs in geometry. Recently, they have been working on developing their own conjectures and then proving their conjectures rigorously.

(1) Lee: We've constructed three parallelograms that have *A*, *B*, and *C*, as vertices: *ADCB*, *AEBC*, and *ABFC*. Now we want to prove two things about the new points in our picture: (1) that the three new points—*D*, *E*, and *F*—form a large triangle and (2) that our original points—*A*, *B*, and *C*—are midpoints of the sides of triangle *DEF*.



(2) Chris: Well, the first part is easy. *Any* three points form a triangle.

(3) Matei: Um, as long as they don't all lie on the same line. Hmm…

(4) Chris: Well, we can *see* that they don't.

(5) Matei: In this case, but that might *not* be so easy to prove for all cases.

(6) Chris: So, let's start with Lee's second part. We'll just assume that sides *DE*, *EF*, and *FD* are straight and make a triangle. So what is it that we want to prove? Oh, that *A*, *B*, and *C* are the midpoints of those sides.

(7) Lee: We created *D*, *E*, and *F* to form parallelograms with *A*, *B* and *C*. So, the proof will have to have something to do with properties of parallelograms. To get our parallelograms, we made the opposite sides congruent and parallel. For example, in parallelogram *ADCB*, segment *AD* is congruent and parallel to segment *BC*.

(8) Matei: But in parallelogram *AEBC*, we can see that…

(9) Chris: Oh, right! Line segment *EA* is *also* congruent and parallel to line segment *BC* since they are opposite sides of parallelogram *AEBC*.

(10) Lee: So if *both AD and EA* are congruent to *BC*, then they are congruent to each other! (Isn't that called the transitive property?) Ta da! Point *A is* the midpoint of line segment *DE*.

(11) Matei: I agree that *AD* is congruent to *AE*, but we still don't know if points *E, A,* and *D* form a straight line so we can't say point *A* is the midpoint of line segment *DE* yet. Can we show that those three points are collinear?

(12) Chris: Well, we also know that *AD* and *AE* are both parallel to *BC*. And *AD* and *AE* have point *A* in common. Doesn't that mean the two form a straight line?

(13) Lee: Does it? Hmm. That does make sense to me but I'm not sure if that's enough to *prove* that *D*, *A*, and *E* form a line.

(14) Chris: Well, what else do we know about parallel lines besides the fact that they don't touch?

(15) Lee: You know what this makes me think of? Remember when we learned to construct a parallel line in dynamic geometry software? We chose a line and a point, and that was enough. There was only one parallel line through that point. We just took that for granted because it seems like it just has to be true. How could there be any other parallel line through that point? But does mathematics say there *must* always be *exactly one* parallel line or is that just how this particular piece of software was built?

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. In line 15 of the dialogue, Lee says:

   > There was only one parallel line through that point. We just took that for granted…. But does mathematics say there *must* always be *exactly one* parallel line […]?

   How would you respond to a student in your class who asked that question?

3. In line 12, Chris suggests the structure of a proof that $\overline{AD}$ and $\overline{AE}$ lie on the same line. Is this proof complete? If not, what is missing?

4. How might a geometry student, unfamiliar with the parallel postulate, go about proving that points *D, A,* and *E* are collinear? In what formats might such a proof legitimately be written?

5. How can multiple approaches to a proof be handled during a whole-class discussion?

6. What are some challenges students may encounter when trying to write a proof?

7. If we choose three points *A*, *B*, and *C* arbitrarily, can we *always* find three points *R*, *S*, and *T* such that each is the fourth vertex of a parallelogram whose other three vertices are *A*, *B*, and *C*?

8. If we choose three non-collinear points *A*, *B*, and *C* arbitrarily, and each point *R*, *S*, and *T* is the fourth vertex of a parallelogram whose other three vertices are *A*, *B*, and *C*, is it *always* the case that *A*, *B*, and *C* will be (in some order) midpoints of the sides of $\triangle RST$? Why or why not?

9. Draw an arbitrary quadrilateral and the midpoint of each side. Connect each midpoint to its adjacent sides' midpoints. What shape is produced? Prove your conjecture.

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
Make sense of problems and persevere in solving them.	This dialogue illustrates two aspects of MP 1: "analyz[ing] givens, constraints, relationships, and goals," and puzzling out an "entry point." Lee (line 1) frames the problem by stating what they know and what they are trying to prove. Matei makes a good catch on Chris's assumption (line 3) when adding the constraint "as long as they don't all lie on the same line." Abandoning the harder task (lines 5–6) should not be interpreted as "not persevering." Making a conscious choice to follow up easier leads might well give one the results needed for the harder tasks and should be viewed as one way students can look for an entry point to a problem. In lines 6–10, the students use previously established results—their proof that the points they've found do create parallelograms along with *A, B,* and *C*—and their knowledge about parallelograms to generate the outline of a proof that point *A* is the midpoint of $\overline{DE}$. They recognize what part of that proof is left unfinished (line 11). And they have made "conjectures about the form…of the solution and [have planned] a solution pathway" (MP 1).
Construct viable arguments and critique the reasoning of others.	Matei's realization (line 5) that one must *prove* that points *D, E,* and *F* aren't collinear represents a further attempt to make complete sense of the problem (MP 1) and a recognition of the need for a "logical progression of statements to explore the truth of their conjectures" (MP 3). *Finding* the proof—a logical train of steps from what one knows (the givens) to the conclusion—is a puzzle. *Articulating* the proof requires clarity and precision, the focus of MP 6.
Attend to precision.	Matei's specification (line 3) that *not* all triples of points make a triangle adds precision to the original conjecture. The process of articulating the proof precisely aids in building the logic of the argument, thereby supporting the development of the proof.

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Commentary on the Mathematics

About proofs

The most prominent elements of mathematical practice in this kind of proof activity are "analyz[ing] givens, constraints, relationships, and goals" (MP 1); puzzling out an "entry point" (also listed as MP 1); and all the listed aspects of MP 3, constructing viable arguments. In the course of making any argument clear, one must also attend to precision (MP 6). *Almost any* proof activity is likely to require all three of these elements of mathematical practice.

Proof is a good example of a mathematical situation in which there is no "rule" for a solution path. We must look for places to start, end goals, and possible connections. And we must often work in both directions toward the middle. Having chosen a potential starting place, we look for facts we know might move us a step toward the goal, but we can't immediately know whether that starting place will suffice or be fruitless. We also look for facts that can help move us a step backward from the known goal. The complexity and lack of a formulaic approach to proof is one reason why proof can be perceived as the off-putting "hard stuff," and also the ultimate satisfying game when one succeeds.

Paragraph proof vs. two-column proof

While gaps in a proof can leave statements with no obvious justification—potentially being just leaps of faith—standard mathematical style often assumes that the reader has some common background with the writer, and so it omits (or only lightly reminds the reader of) what are obvious algebraic steps or the most basic definitions. Students are sometimes taught to present proofs in a form that itemizes even the most minute steps. This is sometimes justified as teaching care and systematic building of an argument, but a risk is that the minutiae distracts from the structure of the argument and makes proof look like an arbitrary exercise, not an attempt to construct a viable argument and communicate it clearly. In secondary schools—not in college or graduate or professional mathematics—this more-meticulous form of proof is often found formatted in two columns. Outside of secondary schools, paragraphs proofs are the norm.

Parallel postulate

A key idea in this dialogue is Euclid's fifth postulate, the parallel postulate. Without explicitly citing it, Lee states its implications in line 15 when saying "There was only one parallel line through that point." In fact, Lee's comments were particularly insightful about the very nature of a postulate: "We just took that for granted because it seems like it just has to be true. How could there be any other parallel line through that point?" (line 15). Postulates are the starting places that we "take for granted"—self-evident truths that cannot be proven from more basic axioms.

This particular postulate, the parallel postulate, is essential for all results in geometry that depend on parallel lines, such as 1) if two parallel lines are crossed by a transversal, their alternate interior angles are congruent; and the less obviously connected fact that 2) the sum of the angles in a triangle (on a plane) is 180°. Many current geometry courses do not develop results as a systematic construction starting from the postulates, and so the parallel postulate (and others) may be less familiar to students than properties of shapes (including the sum of angles in a triangle, the proof of which depends on the parallel postulate!).

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Evidence of the Content Standards

In the dialogue students are trying to prove two conjectures about the diagram they drew; namely that the six points lie on a triangle (G-CO.C.10) and that the three original vertices are midpoints (G-CO.C.9). Proving these conjectures regarding triangles and midpoints involves the use of various definitions, postulates, and theorems.

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In lines 7–9 of the dialogue, students claim that $\overline{AD}$ and $\overline{AE}$ are parallel and congruent to $\overline{BC}$. What justifies this claim?

2. Name two more sets of line segments that are parallel and congruent.

3. In line 12, Chris tries to explain why $\overline{AD}$ and $\overline{AE}$ lie on the same line. Is this proof complete? If not, what is missing?

Related Mathematics Tasks

1. Write a proof to show that that points *D, A,* and *E* are collinear.

2. How would you finish writing the proof that the six points all form a large triangle in which points *A, B,* and *C* are midpoints of the three sides?

3. For different coordinates of points *A, B,* and *C* with possible fourth vertices of a parallelogram labeled *D, E,* and *F*, would you still get a large triangle when connecting points *D, E,* and *F*, whose sides have midpoints *A, B,* and *C*? Why or why not?

4. The Midline Theorem starts that the line segment (midline) connecting the midpoints of two sides of a triangle is half the length and parallel to the third side. Prove this theorem.

5. Draw an arbitrary quadrilateral and the midpoint of each side. Connect each midpoint to its adjacent side's midpoint. What shape is produced? Prove your conjecture.

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. In line 15 of the dialogue, Lee says:
   There was only one parallel line through that point. We just took that for granted…. But does mathematics say there *must* always be *exactly one* parallel line […]?

   How would you respond to a student in your class who asked that question?

   A good response would, of course, depend on what you believe is most likely to feed that student's interests and understanding. The mathematical truth has two parts. *In the plane*, which is the context of this investigation, the simple answer is yes, we do "take it for granted"—which is to say, we postulate—that given a line ℓ and a point P, there is exactly one line parallel to ℓ through P. This is Euclid's fifth postulate.

   But there are also sensible consistent geometries that apply to surfaces other than the plane. For example, imagine walking perfectly straight—veering neither right nor left—on the surface of a sphere. Eventually, we return to where we started (in fact straight lines on a sphere trace out a great circle—one whose center is the same as the center of the sphere). *Any other perfectly straight path* on that sphere will intersect with our path, so there can be no parallel lines—no two straight paths that do not intersect—on a sphere. There also exist surfaces on which infinitely many straight paths through a point may not intersect a given straight path. So, the slightly more complicated answer is that mathematics does not say that "there must always be exactly one parallel line"; that claim, and all claims that make use of it (for example, the 180° in a triangle), is true for the plane, but only for the plane.

3. In line 12, Chris suggests the structure of a proof that $\overline{AD}$ and $\overline{AE}$ lie on the same line. Is this proof complete? If not, what is missing?

   Chris's proof has all the right elements—two segments, each parallel to the same line, and through a common point—but doesn't provide the foundation for the conclusion that those two segments lie on a single line and, therefore, so do their endpoints. Chris might refer to Euclid's fifth postulate, the parallel postulate, that, on a plane, exactly one line through a given point is parallel to a given line. That, by itself, clinches the proof. This means since $\overline{AE}$ and $\overline{DA}$ are both parallel and congruent to $\overline{CB}$ (opposite sides of parallelograms are parallel and congruent), and according to the parallel postulate there is only one line through point A parallel to $\overline{CB}$, which must mean that $\overline{AE}$ and $\overline{DA}$ both lie on the same line. (A similar argument can also be made to show that $\overline{DC}$ and $\overline{CF}$ lie on the same line as well as $\overline{EB}$ and $\overline{BF}$.)

For students who have not experienced building their geometry synthetically from the postulates, they might use the (equivalent) notion, taught in algebra: on the plane, a given point and slope defines a unique line, and lines with the same slope are parallel. If the problem were presented on the coordinate grid, students could argue that $\overline{AD}$ and $\overline{AE}$ have the same slope (both parallel to $\overline{BC}$) and both pass through point A, so they must lie on the same line

4. How might a geometry student, unfamiliar with the parallel postulate, go about proving that points *D, A,* and *E* are collinear? In what formats might such a proof legitimately be written?

Geometry students are sometimes less familiar with Euclid's postulates than with the fact that angles on a straight line add up to 180°. Using this fact, a formal proof in prose ("paragraph form") could be as simple as saying:

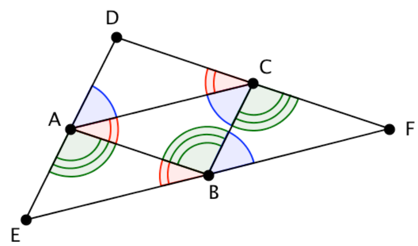
Because $\overline{AE} \parallel \overline{BC}$ with transversal $\overline{AB}$ and $\overline{AD} \parallel \overline{BC}$ with transversal $\overline{AC}$, several pairs of congruent alternate interior angles can be identified. Specifically, $\angle EAB \cong \angle ABC$ and $\angle DAC \cong \angle ACB$. Also, since the sum of a triangle's interior angles is equal to 180°, we now know that $m\angle EAB + m\angle BAC + m\angle DAC = 180^{\circ}$ as well.

Another format for writing proofs is the two-column proof, often used in secondary schools. Below is an example:

Statement	Reason
1. Points *D, A, B,* and *C* form a parallelogram. Points *E, A, C,* and *B* form a parallelogram.	1. Given
2. $\overline{AE} \parallel \overline{CB}$ and $\overline{DA} \parallel \overline{CB}$	2. Definition of a parallelogram
3. $\overline{AC}$ is a transversal across $\overline{DA}$ and $\overline{CB}$. $\overline{AB}$ is a transversal across $\overline{AE}$ and $\overline{CB}$.	3. Definition of a transversal
4. $\angle EAB \cong \angle ABC$ and $\angle DAC \cong \angle ACB$	4. Alternate interior angles are congruent for parallel lines cut by a transversal

5. $\angle CAB \cong \angle CAB$	5. Reflexive property of congruence
6. $m\angle BCA + m\angle ABC + m\angle CAB = 180^\circ$	6. The sum of the interior angles in a triangle is 180°
7. $m\angle DAC + m\angle EAB + m\angle CAB = 180^\circ$	7. Substitution
8. Points *D, A,* and *E* lie on the same line and are collinear.	8. A line has 180°

Note: A similar argument can be made to show that points *D, C,* and *F* and points *E, B,* and *F* are collinear. Since we have parallelograms *AEBC*, *DABC* and *ABFC*, we can identify six sets of parallel line segments crossed by a transversal resulting in three sets of congruent alternate interior angles (see figure below). The diagram shows that the three angles that surround any one of the points *A, B,* or *C* are congruent to the interior angles of $\triangle ABC$, so just like the interior angles of a triangle, they must add up to 180 degrees.



5. How can multiple approaches to a proof be handled during a whole-class discussion?

   Often, several possible proofs can be given, and we would hope that students will share multiple approaches. It can be helpful to record all ideas as students discuss them. When the class later reflects upon the recorded ideas, they might compare the approaches and see what ideas offer significantly different proofs for the same statement and what ideas might belong to the same proof.

6. What are some challenges students may encounter when trying to write a proof?

   Students sometimes 1) don't realize all the properties, postulates, and theorems they have available and 2) have difficulty figuring out a pathway to the statement they are trying to prove. One way to help students overcome these challenges is by providing them with lots of experience in writing proofs and making proofs a central component in the curriculum throughout grade levels, not just a unit that occurs in geometry class.

7. If we choose three points *A*, *B*, and *C* arbitrarily, can we *always* find three points *R*, *S*, and *T* such that each is the fourth vertex of a parallelogram whose other three vertices are *A*, *B*, and *C*?

   Only if points *A*, *B*, and *C* are non-collinear will they make a triangle, and so only in that circumstance can a fourth point be chosen as the fourth vertex of a (non-degenerate) parallelogram. But if points *A*, *B*, and *C* are non-collinear, then a suitable *R*, *S*, and *T* can always be chosen.

8. If we choose three non-collinear points A, B, and C arbitrarily, and each point R, S, and T is the fourth vertex of a parallelogram whose other three vertices are A, B, and C, is it *always* the case that A, B, and C will be (in some order) midpoints of the sides of $\triangle RST$? Why or why not?

   Yes. The proof given in question 4 was not based on any specific choice of points A, B, and C other than that they be non-collinear.

9. Draw an arbitrary quadrilateral and the midpoint of each side. Connect each midpoint to its adjacent sides' midpoints. What shape is produced? Prove your conjecture.

   The shape produced when the four midpoints of a quadrilateral are connected is a parallelogram. To prove this conjecture, draw a quadrilateral with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$ and their respective midpoints E, F, G, and H. Connecting the midpoints you get quadrilateral $EFGH$. Next draw the diagonals $\overline{AC}$ and $\overline{BD}$. $\overline{EH}$ and $\overline{FG}$ are the midlines of $\triangle ABD$ and $\triangle BCD$, respectively. By the midline theorem, those two segments are parallel to $\overline{BD}$ and, therefore, parallel to each other. Similarly, $\overline{EF}$ and $\overline{HG}$ are midlines of $\triangle BAC$ and $\triangle DAC$; by the same argument, $\overline{EF}$ and $\overline{HG}$ are parallel to each other. Because the opposite sides of quadrilateral $EFGH$ are parallel, then it is a parallelogram by definition.

   It may be helpful to use dynamic geometry software to explore this problem. For example, use the applet (link below) to explore the quadrilateral $ABCD$ (in black) and the parallelogram $EFGH$ (in red) formed by its midpoints. The diagonals of the quadrilateral $ABCD$ are also shown as dotted lines. Link: http://www.geogebratube.org/material/iframe/id/19933/width/550/height/400/border/888888/rc/false/ai/false/sdz/false/smb/false/stb/false/stbh/true

   If using this example with students, encourage students to experiment with the construction before moving to the proof. Often, the key insight in the proof could emerge from the experiment. This would be a great opportunity to demonstrate how viable arguments and reasoning can be used as a research technique, not just as a method for establishing conviction.

Possible Responses to Student Discussion Questions

1. In lines 7–9 of the dialogue, students claim that $\overline{AD}$ and $\overline{AE}$ are parallel and congruent to $\overline{BC}$. What justifies this claim?

   We are given that $DABC$ is a parallelogram. $\overline{AD}$ is parallel and congruent to $\overline{BC}$ because they are opposite sides of that parallelogram. For the same reason, $\overline{AE}$ and $\overline{BC}$ are parallel and congruent: they are opposite sides of parallelogram $AECB$.

2. Name two more sets of line segments that are parallel and congruent.

   Sets of line segments that are parallel and congruent are $\overline{FC}$ and $\overline{AB}$, $\overline{DC}$ and $\overline{AB}$, $\overline{BF}$ and $\overline{AC}$, and $\overline{EB}$ and $\overline{AC}$.

3. In line 12, Chris tries to explain why $\overline{AD}$ and $\overline{AE}$ lie on the same line. Is this proof complete? If not, what is missing?

   Chris's proof has all the right elements—two segments, each parallel to the same line, and through a common point—but doesn't provide the foundation for the conclusion that those two segments lie on a single line and, therefore, so do their endpoints. Chris might refer to Euclid's fifth postulate, the parallel postulate, that on a plane, exactly one line through a given point is parallel to a given line. That, by itself, clinches the proof.

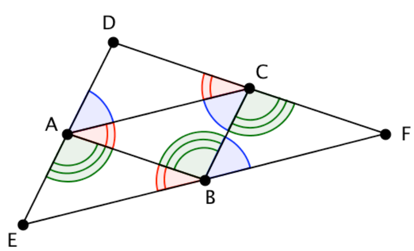
   For students who have not experienced building their geometry synthetically from the postulates, they might use the (equivalent) notion, taught in algebra, that a given point and slope on the plane defines a unique line, and that lines with the same slope are parallel. If the problem had been presented on the coordinate grid, students could have argued that $\overline{AD}$ and $\overline{AE}$ have the same slope (both parallel to $\overline{BC}$) and both pass through point *A*, so they must lie on the same line.

Possible Responses to Related Mathematics Tasks

1. Write a proof to show that that points *D, A,* and *E* are collinear.

   Using the parallel postulate:
   Quadrilaterals *AEBC* and *DABC* are parallelograms and, thus, have opposite sides that are parallel and congruent. This means $\overline{AE}$ and $\overline{DA}$ are both parallel and congruent to $\overline{CB}$. The parallel postulate states that there is only one line through point *A* parallel to $\overline{CB}$; thus, $\overline{AE}$ and $\overline{DA}$ must lie on the same line, making points *E, A*, and *D* collinear.

   Using 180° *in a line:*
   Given that *AEBC* and *DABC* are parallelograms, we can identify sets of parallel segments crossed by a transversal (e.g., $\overline{AE} \parallel \overline{CB}$ with transversal $\overline{AB}$; likewise, $\overline{DA} \parallel \overline{CB}$ with transversal $\overline{AC}$) and, therefore, resulting sets of congruent alternate interior angles (see figure below):



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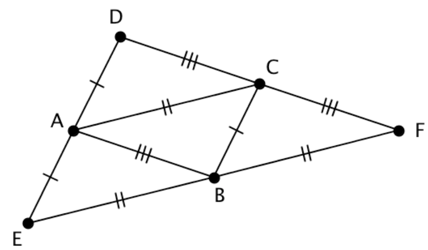
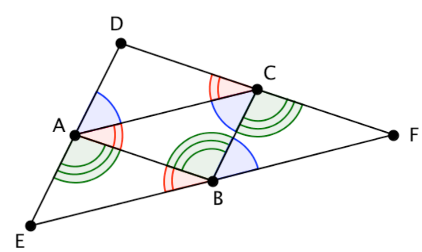
The diagram shows that the three angles that surround any one of the points *A, B,* or *C* are congruent to the interior angles of $\triangle ABC$. Since the sum of the interior angles of a triangle is 180°, so is the sum of the marked angles surrounding point *A* (or *B* or *C*): $m\angle EAB + m\angle BAC + m\angle DAC = 180^\circ$;therefore, $\angle EAD$ is a straight angle and *E*, *A*, and *D* are collinear. Because parallelograms have congruent opposite sides, point *A* is the midpoint of $\overline{DE}$.



2. How would you finish writing the proof that the six points all form a large triangle in which points *A, B,* and *C* are midpoints of the three sides?

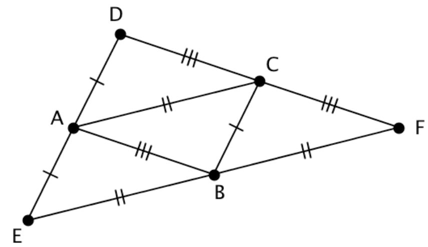
Using the parallel postulate:
Quadrilaterals *AEBC* and *DABC* are parallelograms and thus have opposite sides that are parallel and congruent. This means $\overline{AE}$ and $\overline{DA}$ are both parallel and congruent to $\overline{CB}$. The parallel postulate states that there is only one line through point *A* parallel to $\overline{CB}$, thus $\overline{AE}$ and $\overline{DA}$ must lie on the same line, making points *E, A,* and *D* collinear. Moreover, because $\overline{AE}$ and $\overline{DA}$ are congruent, *A* is the midpoint of $\overline{ED}$. The same reasoning shows that because $\overline{DC}$ and $\overline{CF}$ are both parallel and congruent to $\overline{AB}$, points *D, C,* and *F* are collinear, and *C* is the midpoint. The same applies to *F, B,* and *E*.

Using 180° *in a line:*
Given that *AEBC*, *DABC*, and *ABFC* are parallelograms, we can identify six sets of parallel segments crossed by a transversal (e.g., $\overline{AE} \parallel \overline{CB}$ with transversal $\overline{AB}$; likewise, $\overline{DA} \parallel \overline{CB}$ with transversal $\overline{AC}$) and three resulting sets of congruent alternate interior angles (see figure below): $\angle EAB \cong \angle CBA \cong \angle BCF$ and $\angle DAC \cong \angle BCA \cong \angle CBF$ and $\angle ABE \cong \angle BAC \cong \angle DCA$



The diagram shows that the three angles that surround any one of the points *A, B,* or *C* are congruent to the interior angles of $\triangle ABC$. Since the sum of the interior angles of a triangle is 180°, so is the sum of the marked angles surrounding point *A* (or *B* or *C*):

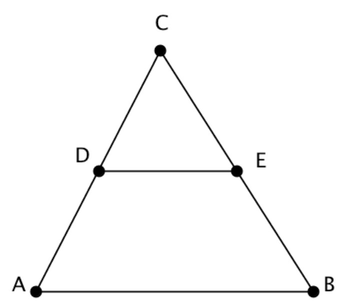
$m\angle EAB + m\angle BAC + m\angle DAC = 180^{\circ}$ and, therefore, $\angle EAD$ is a straight angle, thus making *E*, *A*, and *D* collinear. The same reasoning proves collinearity of points *E*, *B,* and *F* and of points *D*, *C* and *F*. Because parallelograms have congruent opposite sides, points *A*, *B,* and *C* are midpoints of their respective triangle sides (see figure below).



3. For different coordinates of points *A, B,* and *C* with possible fourth vertices of a parallelogram labeled *D, E,* and *F*, would you still get a large triangle when connecting points *D, E,* and *F*, whose sides have midpoints *A, B,* and *C*? Why or why not?

   Yes, as long as points *A*, *B* and *C* are non-collinear. Regardless of the coordinates of points *A*, *B* and *C*, you would get a large triangle with midpoints at *A*, *B* and *C* when you connect the possible parallelogram's fourth vertex. The reason why you would get the same result can be seen by looking at the proof in the previous question. The proof was not based on any specific coordinates but was done generally. Since the result was proven generally, that means it works for any particular instance (i.e., coordinates of points *A*, *B*, and *C*).

4. The Midline Theorem starts that the line segment (midline) connecting the midpoints of two sides of a triangle is half the length and parallel to the third side. Prove this theorem.



   We know that points *D* and *E* are midpoints of $\overline{CA}$ and $\overline{CB}$ respectively, which means $\overline{CD} \cong \overline{DA}$ and $\overline{CE} \cong \overline{EB}$. This implies that $\overline{CA}$ is twice the length of $\overline{CD}$, and $\overline{CB}$ is twice the length of $\overline{CE}$. Since both $\triangle DCE$ and $\triangle ACB$ share $\angle DCE$, that means you can use the SAS Similarity Theorem to show $\triangle DCE \sim \triangle ACB$. Since the triangles are similar, that means all the sides are scaled by the same factor, which in this case is 2, so $\overline{DE}$ is half the length of $\overline{AB}$. Furthermore, since the triangles are similar, all corresponding angles are congruent. Since $\angle CDE \cong \angle DAB$, you can use the AIP Theorem to show that $\overline{DE} \parallel \overline{AB}$.

Proof with Parallelogram Vertices

5. Draw an arbitrary quadrilateral and the midpoint of each side. Connect each midpoint to its adjacent side's midpoint. What shape is produced? Prove your conjecture.

   The shape produced when the four midpoints of a quadrilateral are connected is a parallelogram. To prove this conjecture, draw a quadrilateral with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ and their respective midpoints *E, F, G,* and *H*. Connecting the midpoints, you get quadrilateral *EFGH*. Next, draw the diagonals $\overline{AC}$ and $\overline{BD}$. $\overline{EH}$ and $\overline{FG}$ are the midlines of $\triangle ABD$ and $\triangle BCD$, respectively. By the Midline Theorem, those two segments are parallel to $\overline{BD}$ and, therefore, parallel to each other. Similarly, $\overline{EF}$ and $\overline{HG}$ are midlines of $\triangle BAC$ and $\triangle DAC$; by the same argument, $\overline{EF}$ and $\overline{HG}$ are parallel to each other. Because the opposite sides of quadrilateral *EFGH* are parallel, it is a parallelogram by definition.

   It may be helpful to use dynamic geometry software to explore this problem. For example, use the applet (link below) to explore the quadrilateral *ABCD* (in black) and the parallelogram *EFGH* (in red) formed by its midpoints. The diagonals of the quadrilateral *ABCD* are also shown as dotted lines. Link: http://www.geogebratube.org/material/iframe/id/19933/width/550/height/400/border/888888/rc/false/ai/false/sdz/false/smb/false/stb/false/stbh/true