Rational Exponents

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Rational Exponents* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the value of expressions with rational exponents. The students use their understanding of positive integer exponents as repeated multiplication "steps" to make sense of what a fractional multiplicative step is (i.e., rational exponents).

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 8–10

Target Content Domain: The Real Number System (Number and Quantity Conceptual Category)

Highlighted Standard(s) for Mathematical Content

N-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

Math Topic Keywords: exponents, rational exponents, rules of exponents

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

What is $64^{\frac{1}{2}}$? What about $64^{\frac{1}{3}}$?

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students are studying exponents. They have just learned the meaning of negative exponents and are now figuring out what rational exponents are.

(1) Chris: We just finished figuring out negative exponents. Now we've gotta do *FRACTIONS*???? What *is* $64^{\frac{1}{2}}$ anyway?

(2) Lee: What do you mean?

(3) Chris: Well, I know what 64^3 is and what 64^4 is, but what does it mean when there is a fraction in the exponent?

(4) Lee: Well, what would you *want* $64^{\frac{1}{2}}$ to mean? We know that $\frac{1}{2}$ is between 0 and 1. And we know that 64^0 is 1 and 64^1 is 64. So $64^{\frac{1}{2}}$ must be between 1 and 64.

(5) Chris: *(sarcastically)* Well, that narrows it right down, doesn't it!?! And if it's about what I *want*, I could go for an ice cream cone right now.

(6) Matei: So… since $\frac{1}{2}$ is halfway between 0 and 1, wouldn't $64^{\frac{1}{2}}$ be halfway between 1 and 64? That would be, um, thirty-two! Right?

(7) Chris: You mean $31\frac{1}{2}$, but that can't be right, anyway. You're saying that $64^{\frac{1}{2}}$ is the same as $64\times\frac{1}{2}$, but that's not what exponents do. 5^3 is not even close to 5×3. It's $5\times 5\times 5$. But I still don't know what to do with $64^{\frac{1}{2}}$. I guess I don't really get how these exponents work. They're like multiplication, but they're also not like multiplication. The numbers grow so quickly.

(8) Lee: Yeah, like if I multiply by 10, I can get from 1 to 100 in only 2 steps. Start at 1, multiply by 10, then 10 again.

$10^0 = 1$; I started at 1 and didn't multiply by 10 at all
$10^1 = 10$; I multiply by 10 once, my first step.
$10^2 = 100$; I multiplied by 10 a second time, Voila!

Exponents just show the number of multiplications.

(9) Chris: I *know* that, but *this* problem is $64^{\frac{1}{2}}$! What could that possibly mean?! How do you start at 1 and then multiply by 64 *half of a time*?!

[long pause while the three students think]

(10) Lee: Hmmm…… Well, I was thinking that we need to come up with something that does make sense. So I experimented with my numbers. If we want to get from 1 to 100 in *one* step, we multiply by 100. If the one step is "multiply by 100," then one-half of a step would be "multiply by whatever," and it would take *two* of those "multiply by whatevers" to get us to 100. Right? So, I guess, using multiplication, "one-half of the way" to 100 is 10.

(11) Matei: No way. Half of the way from 1 to 100 is 50. Or, well, $49\frac{1}{2}$, or whatever. No?

(12) Chris: Well, yeah, IF you use *addition* in each of those steps. But I get what Lee is saying. The "steps" that exponents count are all multiplication, so it's what Lee said. If you start at 1 and each "step" is "multiply by 10," then *one* step gets you to 10, the *second* gets you to 100. So…. What if you start at 1 and each step is multiply by 100? If *one* step is multiply by 100, then *half* a step must be multiply by 10.

(13) Lee: So, "half of the way" from 1 to 100, *using multiplication*, would be like saying $100^{\frac{1}{2}}$? So $100^{\frac{1}{2}} = 10$?

(14) Matei: Yuck. I still don't like you saying "half the way," but now I get what you mean. Reasonable enough. OK…. So now I get why $64^{\frac{1}{2}}$ isn't 32, but how does your idea help us *find* $64^{\frac{1}{2}}$?

(15) Chris: Got it! We need a "whatever" to multiply twice to get from 1 to 64.

(16) Matei: Right. We're looking for a number that we can multiply by twice to get from 1 to 64. Isn't that 8? Right? Because $1 \times 8 \times 8$ is 64.

(17) Chris: By George, I think you've got it! So $64^{\frac{1}{2}} = 8$, done!

(18) Lee: Not yet, what about $64^{\frac{1}{3}}$?

(19) Matei: We can think about it the same way. But now, since the exponent is $\frac{1}{3}$, we need something that will take 3 steps instead of 2.

(20) Lee: It has to be smaller than 8…what about 5? $5 \times 5 = 25$, then 25×5 is 125… No, much too big. Oh, besides, that was a silly guess anyway! Five is not a factor of 64, so multiplying by 5 will never get me there.

(21) Chris: So it's really small! Like 2? No, 2^3 is 8. So, not that small.

(22) Lee: So whatever $64^{\frac{1}{3}}$ is, if you cube it, you get 64. What cubed is 64?

(23) Matei: 4

(24) Lee: So a $\frac{1}{3}$ power is just the cube root of the number? So if $x = 64^{\frac{1}{3}}$, then $x^3 = 64$, and $x = \sqrt[3]{64}$.

(25) Matei: So now we can do that for any number. Like $5^{\frac{1}{3}}$ is $\sqrt[3]{5}$. But I like thinking about going "a third of the way" with multiplication, because then you can kind of figure out how big it's supposed to be.

(26) Chris: Yup, 2 is too big for the $\sqrt[3]{5}$ because 2^3 is already 8. Maybe 1.5?

(27) Lee: 1.5^2 is 2.25. Then 1.5 times 2.25 is three point something… Nah, that's too small. Well, at least we know $\sqrt[3]{5}$ is between 1.5 and 2, then.

(28) Chris: I wonder how this works for $64^{\frac{2}{3}}$…

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. What mathematical reasoning are students using when they say that $64^{\frac{1}{2}}$ is between $64^0 = 1$ and $64^1 = 64$?

3. In line 7 of the dialogue, Chris makes the argument that $64^{\frac{1}{2}}$ can't be the same thing as $64 \times \frac{1}{2}$. Is there *any* number for which that number raised to the one-half power is the same as multiplying the original number by one half?

4. In this dialogue, students are trying to make sense of $64^{\frac{1}{2}}$ yet they spend a lot of time discussing integer powers of 10. What good reason might there be for such a conversation to have shifted to these numbers?

5. In line 10, Lee shifts from thinking about powers of 10 to thinking about powers of 100. How does this help the students get to their understanding of what it means to raise a number by a third?

6. How could the rules of exponents be used to find the value of $64^{\frac{1}{3}}$?

7. In line 24, Lee equates $64^{\frac{1}{3}}$ with $\sqrt[3]{64}$ by stating that "if $x = 64^{\frac{1}{3}}$, then $x^3 = 64$" and concludes that $x^3 = 64$ means that *x* must be $\sqrt[3]{64}$ (which is 4). Lee's reasoning is entirely sound and the answer correct, given what a student in Algebra I is likely to know. But one step in Lee's reasoning contains a subtle assumption that the student later learns is not correct. Explain.

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Make sense of problems and persevere in solving them.	Students' arguments in this dialogue are based on sense making, using numerical examples with integer exponents, and trying to preserve the "behavior" (MP 7) that they induce from their examples as they extend the examples to use $\frac{1}{3}$ as an exponent. Chris, Lee, and Matei are trying to make sense out of a subtle and slippery situation. Lee's question (line 4), "What would you *want* $64^{\frac{1}{2}}$ to mean?" is more appropriate than a high school student is likely to realize. When Chris asks (line 9) what it could possibly mean to "multiply by 64 *half of a time*," the student has nailed the problem: the meaning is not something decided by an authority, or even something to "discover," but rather, it is something to *invent*. There is no "natural" meaning for $64^{\frac{1}{2}}$ the way there is for 5^7 or any other exponentiation involving only the natural numbers. We must *give* meaning to this expression if we want it to have a meaning. We're free to assign it any meaning at all, but we want to pick a meaning that is consistent with the natural meaning and behaves in the same way. (See "Commentary on the Mathematics" section below for more information.)
 Look for and make use of structure.	The students' attempts to preserve the way exponents work in this dialogue comprise an informal case of observing and using/preserving structure. These students are using analogy and intuitive arguments, and never achieve a precisely articulated algebraic statement—the explicit use of laws of exponents—but Lee's notion of "steps" (lines 8 and 10) is the informal beginning, using powers of 10 as an example, of what might be expressed more formally as $10^{(\text{step})} \times 10^{(\text{step})} \times 10^{(\text{step})} = 10^{(3\text{ steps})}$, or yet more formally as $(a^b)^3 = a^{3b}$. The students are still trying to make sense of this, numerically, even at the end of the dialogue, but get as far as the informal equivalent of $\left(a^{\frac{1}{3}}\right)^3 = a$.

Construct viable arguments and critique the reasoning of others.	There are several places throughout this exercise where Chris, Lee, and Matei are making conjectures (right or wrong), and then one of the others helps to think it all the way through with counterexamples and careful examination of the idea presented. For example, when Matei says that $64^{\frac{1}{2}}$ must be 32, Chris counters with a critique. Lee (line 8) also very clearly articulates a thought process (MP 6) but, in doing so, creates the argument that supports the student's reasoning.

Commentary on the Mathematics

<u>Extension of operations to a broader domain of numbers</u>

The core of the mathematics here is "extension": taking an operation (exponentiating) whose very definition seems to be rooted in the domain of counting numbers and trying to come up with a consistent interpretation of it over a broader domain of numbers. The dialogue ends with students' logic extending the sensible use of exponents only to unit fractions, but that logic is adequate for all rational numbers.

<u>Reasoning by continuity</u>

Lee argues that since $\frac{1}{2}$ is between 0 and 1, then $64^{\frac{1}{2}}$ must be between 64^0 and 64^1, which the student recognizes as 1 and 64. These numbers are then the upper and lower bounds on what the value of $64^{\frac{1}{2}}$ could be. Similarly, the students reason algebraically that *if* numbers like $64^{\frac{1}{3}}$ and $\sqrt[3]{64}$ exist, then they must behave in a certain way (or the system becomes unpleasantly inconsistent). In this case, the real number that satisfies their logic happens to be an integer.

But they also think about $5^{\frac{1}{3}}$ and $\sqrt[3]{5}$, which is not an integer. Again, they set upper and lower bounds, reasoning that it's between 1.5 and 2 and that they could, with work, squeeze those bounds closer and closer, zooming in on the number line to make better and better approximations. When we see that $1.7^3 < 5$ and $1.8^3 > 5$, we know that $\sqrt[3]{5}$ must be between 1.7 and 1.8, so we try some numbers in between. We see that 1.71^3 is already greater than 5, so we now know that $\sqrt[3]{5}$ must be between 1.7 and 1.71. A few more steps and we convince ourselves that if there *is* a number—that is, if there isn't a "hole" in the number line where we're looking—then we're closing in on that number. This reasoning is an encounter with the notion of limits.

But there is one more subtlety here: they are tacitly assuming that such a number *exists*. The notion that the real number line is complete—that there are no "holes" in it—is correct but must remain an assumption; the proof is beyond high school mathematics.

<u>Notes on notation</u>

The students begin their discussion using the idea that a^b means to multiply *b* copies of *a* together. This convention is a very convenient way to begin developing students' understanding of exponents for any value of *a*. However, as is evident in this Illustration and others (see *Extending Patterns with Exponents),* this convention makes sense only when *b* is a positive integer. So, eventually, students must move beyond this elementary understanding of exponents.

Another underlying idea in this dialogue and the topic of rational exponents is whether or not the following is true:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

While we would like them all to be the same, it is important to note that they might not always be. Consider the case where $a = -64$, $m = 2$, $n = 6$. In this case, we have $(-64)^{\frac{2}{6}}$, which should be the same as $(-64)^{\frac{1}{3}} = -4$. However, if we consider $((-64)^2)^{\frac{1}{6}}$, this value is positive. The result then is that we must standardize what is meant by $a^{\frac{m}{n}}$; the convention is that it means the m^{th} power of $a^{\frac{1}{n}}$ or $\left(a^{\frac{1}{n}}\right)^m$. This standardization is necessary only if the exponent is not expressed in simplest terms and if both *m* and *n* are even numbers. There are other extensions that students will eventually encounter and need to make sense of. At this point, what could $(-64)^{\frac{1}{2}}$ mean to students who are beginning their understanding of rational exponents? It is presumably *not* something "between" $(-64)^0$ and $(-64)^1$. Giving meaning to a *negative* base with a non-integer exponent will require a new set of numbers—the complex numbers.

Evidence of the Content Standards

This dialogue focuses entirely on how students can make sense of what rational exponents mean (N-RN.A.1). In particular lines 8 and 10 extend the meaning of exponentiation as repeated multiplication to come up with an explanation of what multiplying by "half a time" means.

Rational Exponents

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. On what reasoning does Lee in the dialogue base the claim that $64^{\frac{1}{2}}$ is between 1 and 64?
2. What does Matei think the value of $64^{\frac{1}{2}}$ is? Why?
3. How does Chris disprove Matei's claim for why $64^{\frac{1}{2}} = 32$?
4. How did Lee come up with $100^{\frac{1}{2}} = 10$?
5. In lines 10 and 11, Lee and Matei argue about what it means to go "half the way." How is each one defining "half the way"? Under each definition, what would halfway from 1 to 225 be?
6. What is more convincing about Lee's notion of what "half of the way" means?
7. What is the value of $64^{\frac{1}{3}}$? How did the students find the value?

Rational Exponents

Related Mathematics Tasks

1. What is $64^{\frac{1}{6}}$? Explain your answer.

2. What is $8^{\frac{1}{2}}$? Explain your answer.

3. What is $7^{\frac{1}{4}}$? Explain your answer.

4. In line 7, Chris makes the argument that $64^{\frac{1}{2}}$ can't be the same thing as $64 \cdot \frac{1}{2}$. By analogy, we can make the claim that $64^{\frac{1}{3}}$ is not the same as $64 \cdot \frac{1}{3}$. Is there ever a time when a number raised to the one-third power *is* the same as multiplying the original number by one third?

5. In this dialogue, students are trying to make sense of rational exponents. For this task, consider how might you calculate a negative exponent? Try to come up with a value for 10^{-3} and support your answer.

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. What mathematical reasoning are students using when they say that $64^{\frac{1}{2}}$ is between $64^0 = 1$ and $64^1 = 64$?

   Students are using two mathematical principles when they make this statement. The first is the principle of continuity—that there are no "holes" in the number line and that somewhere between 64^0 and 64^1 there must be a number $64^{\frac{1}{2}}$. They also know that the exponent $\frac{1}{2}$ is between 0 and 1, so they expect 64 raised to the $\frac{1}{2}$ power to be between 1 and 64. This is the idea of limits. Both of these ideas are important mathematical principles that show up in calculus that can be explored informally in younger grades.

3. In line 7 of the dialogue, Chris makes the argument that $64^{\frac{1}{2}}$ can't be the same thing as $64 \times \frac{1}{2}$. Is there *any* number for which that number raised to the one-half power is the same as multiplying the original number by one half?

   Yes… when $a = 4$ or 0. The solutions to the equation $(a)^{\frac{1}{2}} = \frac{a}{2}$

4. In this dialogue, students are trying to make sense of $64^{\frac{1}{2}}$ yet they spend a lot of time discussing integer powers of 10. What good reason might there be for such a conversation to have shifted to these numbers?

   They shift to powers of 10 since multiplication by 10 is easier to calculate. By focusing on an example that is easy to calculate, the students can focus on the operations and find a structure they can use.

5. In line 10, Lee shifts from thinking about powers of 10 to thinking about powers of 100. How does this help the students get to their understanding of what it means to raise a number by a third?

   The students initially find powers of 10 and have a series of numbers (1, 10, 100) in which they can go from 1 to 100 in 2 multiplicative steps. They switch perspectives to powers of 100 because they want to think about $100^0 = 1$ and $100^1 = 100$ as endpoints in the series of numbers (1, 10, 100). Now, if one large step (exponent of 1) takes you from 1 to 100, each of the two smaller steps would correspond to an exponent of $\frac{1}{2}$. By looking at 1, 10, 100 as powers of 100, they were able to give meaning to $100^{\frac{1}{2}}$ and, therefore, come up with a sensible definition for rational exponents.

6. How could the rules of exponents be used to find the value of $64^{\frac{1}{3}}$?

   The students in this dialogue informally used the rules of exponents to make sense of rational exponents. Below is an explanation using the rule that $(a^m)^n = a^{m \cdot n}$ of why $64^{\frac{1}{3}} = 4$.

$$\left(64^{\frac{1}{3}}\right)^3 = 64^{\frac{1}{3} \cdot 3}$$

$$\left(64^{\frac{1}{3}}\right)^3 = 64^1$$

$$\left(64^{\frac{1}{3}}\right)^3 = 64$$

$$\sqrt[3]{\left(64^{\frac{1}{3}}\right)^3} = \sqrt[3]{64}$$

$$64^{\frac{1}{3}} = \sqrt[3]{64}$$

$$64^{\frac{1}{3}} = 4$$

7. In line 24, Lee equates $64^{\frac{1}{3}}$ with $\sqrt[3]{64}$ by stating that "if $x = 64^{\frac{1}{3}}$, then $x^3 = 64$" and concludes that $x^3 = 64$ means that x must be $\sqrt[3]{64}$ (which is 4). Lee's reasoning is entirely sound and the answer correct, given what a student in Algebra I is likely to know. But one step in Lee's reasoning contains a subtle assumption that the student later learns is not correct. Explain.

   In general, Algebra I students do not yet know about complex numbers, nor do they know the Fundamental Theorem of Algebra, which states that a polynomial of degree *n* has *n* roots (not necessarily all distinct). So, the third degree polynomial that Lee creates, $x^3 = 64$, has not just the one root Lee gives ($x = 4$), but three roots, two of which are complex: $-2+2i\sqrt{3}$ and $-2-2i\sqrt{3}$. The roots can be found by rewriting $x^3 = 64$ as $x^3 - 64 = 0$ and factoring. Since we know that 4 is a root, that means $(x-4)(x^2+4x+16)=0$. Using the quadratic formula to find the roots of the quadratic, we can find the two complex roots.

Possible Responses to Student Discussion Questions

1. On what reasoning does Lee in the dialogue base the claim that $64^{\frac{1}{2}}$ is between 1 and 64?

   In line 4, Lee argues that since $\frac{1}{2}$ is between 0 and 1, $64^{\frac{1}{2}}$ must be between 64^0 and 64^1. Lee knows that $64^0 = 1$ and that $64^1 = 64$, so $64^{\frac{1}{2}}$ must be between 1 and 64.

2. What does Matei think the value of $64^{\frac{1}{2}}$ is? Why?

   Matei argues that the value of $64^{\frac{1}{2}}$ is 32 since half of the distance from 1 to 64 is 32.

3. How does Chris disprove Matei's claim for why $64^{\frac{1}{2}} = 32$?

   Chris realizes that $64^{\frac{1}{2}} = 32$ is similar to saying, "multiply the base by the exponent." Chris explains that raising something to a power can't be the same as multiplying the number by the exponent, since exponents are all about multiplying and they often get larger very quickly (when working with integer bases). The student gives the counterexample of 5^3, in which $5 \cdot 5 \cdot 5 \neq 5 \cdot 3$.

4. How did Lee come up with $100^{\frac{1}{2}} = 10$?

   When looking at powers of 10, Lee notices that it takes two multiplying steps to go from 1 to 100 (1➔10 ➔100). The student then thinks about powers of 100, for which only one step is needed to go from 1 to 100. Lee reasons that if 1 step (100^1) takes you from 1 to 100, "half" of that kind of step must take you to 10, because two such "half-steps" would be two multiplications by 10.

5. In lines 10 and 11, Lee and Matei argue about what it means to go "half the way." How is each one defining "half the way"? Under each definition, what would halfway from 1 to 225 be?

   Lee is defining "half the way" using multiplication. For Lee, a number that is "halfway from 1 to 100" using multiplication is a number that is multiplied twice to the starting number (in this case 1) to make that full step from 1 to 100. Matei is thinking about "50. Or, well, $49\frac{1}{2}$, or whatever," but defining "half the way" using addition. In this case, Matei seems to be looking for a number that can be added twice to get from 1 to 100. For Matei, half of the way is half of the distance on a number line (additive distance). Using Lee's definition, 15 would be half the way from 1 to 225 because $1 \cdot 15 \cdot 15 = 225$. Matei might think of halfway as 112 (because $1 + 112 + 112 = 225$) or 113 (because it is "halfway," the result of adding the first 112).

6. What is more convincing about Lee's notion of what "half of the way" means?

   Lee's definition seems to be more consistent with what students already know about exponentiation, namely that it is based on multiplication. Furthermore, using an additive definition for half the way would lead to inconsistent rules for whole number and rational exponents. It would claim that $5^{\frac{1}{3}} \approx 5 \cdot \frac{1}{3}$ while they already know that $5^3 \neq 5 \cdot 3$ and that, instead, $5^3 = 5 \cdot 5 \cdot 5$.

7. What is the value of $64^{\frac{1}{3}}$? How did the students find the value?

   The value is 4. They know this since $4 \cdot 4 \cdot 4 = 64$. Similarly $64^{\frac{1}{3}}$ is the same as the positive root of $\sqrt[3]{64}$.

Possible Responses to Related Mathematics Tasks

1. What is $64^{\frac{1}{6}}$? Explain your answer.

   $64^{\frac{1}{6}} = 2$ because $\sqrt[6]{64} = 2$; alternatively, $2^6 = 64$.

2. What is $8^{\frac{1}{2}}$? Explain your answer.

   $8^{\frac{1}{2}}$ is not an integer. Because $\left(8^{\frac{1}{2}}\right)^2 = 8$, we can tell that $8^{\frac{1}{2}}$ must be between 2 and 3 (because 2^2 is too little and 3^2 is too big). By "zooming in" a bit further we can see that the value is somewhere between 2.8 and 2.9 since $2.8^2 = 7.84$ (too little) and $2.9^2 = 8.41$ (too big).

3. What is $7^{\frac{1}{4}}$? Explain your answer.

   $7^{\frac{1}{4}}$ is also not a whole number since no whole number raised to the fourth power will equal 7. A calculator can, of course, give us an approximation quickly, but even without using the exponentiation feature, we can experiment to see that $7^{\frac{1}{4}}$ is between 1.6 and 1.7, since $1.6^4 = 6.5536$ and $1.7^4 = 8.3521$.

4. In line 7, Chris makes the argument that $64^{\frac{1}{2}}$ can't be the same thing as $64 \cdot \frac{1}{2}$. By analogy, we can make the claim that $64^{\frac{1}{3}}$ is not the same as $64 \cdot \frac{1}{3}$. Is there ever a time when a number raised to the one-third power *is* the same as multiplying the original number by one third?

   Yes, if your starting number is either 0 or $\pm\sqrt{27}$. Below is an explanation:

$$a^{\frac{1}{3}} = \frac{1}{3}a$$

$$\left(a^{\frac{1}{3}}\right)^3 = \left(\frac{1}{3}a\right)^3$$

$$a^{\frac{1}{3}\cdot 3} = \left(\frac{1}{3}\right)^3 \cdot a^3$$

$$a = \frac{1}{27} \cdot a^3$$

$$27a = a^3$$

$$27 = a^2$$

$$\pm\sqrt{27} = a$$

5. In this dialogue, students are trying to make sense of rational exponents. For this task, consider how might you calculate a negative exponent? Try to come up with a value for 10^{-3} and support your answer.

   $10^{-3} = .001$. Check student work for their explanation. Encourage them to write down several powers of 10 and look for a pattern or structure that they can use to make sense of negative powers. Also, please note there is an Illustration on this topic called *Extending Patterns with Exponents*.