Rectangles with the Same Numerical Area and Perimeter

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Rectangles with the Same Numerical Area and Perimeter* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find all rectangles that have the same numerical area and perimeter. After trying different rectangles of specific dimensions, students finally develop an equation that describes the relationship between the width and length of a rectangle with equal area and perimeter.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 2: Reason abstractly and quantitatively.
MP 7: Look for and make use of structure.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 8–9

Target Content Domain: Creating Equations (Algebra Conceptual Category)

Highlighted Standard(s) for Mathematical Content

A.CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A.CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*

Math Topic Keywords: area, perimeter, rational expressions

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Find the dimensions of all rectangles whose area and perimeter have the same numerical value.

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Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

The students in this dialogue know how to determine the area and perimeter of rectangles, and have experience with manipulating equations. This dialogue comes at a time of year when students are studying rational expressions.

(1) Lee: Let's start by finding at least one that works.

(2) Chris: OK, so let's start with squares. All the sides are the same in squares.

(3) Lee: We don't care that all the sides are the same! We want the numerical value of the ***area*** and the ***perimeter*** to be the same.

(4) Matei: We can still start with squares and see if they work.

(5) Lee: OK. Let's make a table. *[they write]*

Side length of square	Area	Perimeter
1	1	4
2	4	8
3	9	12
4	16	16
5	25	20
6	36	24

(6) Chris: We've got a winner! 4 works. I told you squares could work. Can any other squares work?

(7) Matei: I think that might be the only one. Look, the area is getting bigger much faster than the perimeter when we make the side length bigger. Once we pass the side length of 4, I don't see how the perimeter could ever catch up to the area.

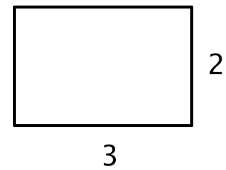
(8) Chris: Area's definitely winning that race! Okay, so you said we had to think about all rectangles, not just squares. So, what's next?

(9) Lee: A table is more complicated for non-squares. There are two different side lengths.

(10) Chris: Yeah… how are you going to make that work?

(11) Lee: Well, let's try at least a few!

(12) Chris: What about 2 and 3? *[draws the following]*



(13) Lee: Perimeter of 10 and area of 6. Doesn't work.

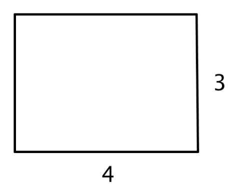
(14) Matei: So, then what about $2\frac{1}{2}$ and 3? That's a perimeter of 11 and area of $7\frac{1}{2}$. That's closer.

(15) Chris: Wait! Don't make it harder, yet! Let's stick with regular numbers and see what happens.

(16) Matei: You mean WHOLE numbers?

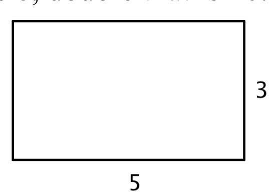
(17) Chris: Yeah. Sure. Whatever.

(18) Lee: Let's try 4 and 3… 4 times 3 is 12. $4+3$ is 7, double that and get 14. *[draws the following]*



(19) Chris: Close!

(20) Lee: 5 times 3 is 15. $5+3$ is 8, double that is 16. *[draws the following]*



(21) Chris: Even closer!

(22) Matei: I don't think it will ever work if the area is odd, because the perimeter has to be even. After all, to find the perimeter, we add the two numbers and double the sum. It *has* to be even.

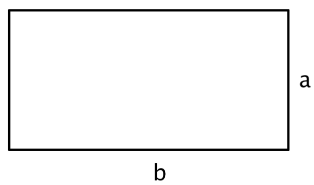
(23) Chris: Well, that's odd!

(24) Lee: Okay, so we were close with 5 and 3, but we need an even product. 6 times 3 is 18. $6+3$ is 9 and double 9 is 18. *[draws the following]*

(25) Chris: Hooray! Another winner! We found one! Is that all of them?

[There is a long pause during which everyone thinks about the problem.]

(26) Matei: I don't know, but I don't think so. What are we doing each time we try an example? *[draws the following]*



If we call the side lengths a and b, then area is a times b. *[writes* $\text{Area} = ab$ *]* To find perimeter, we've been adding a and b and then doubling that sum. *[writes* $\text{Perimeter} = (a+b)\times 2$ *]* Then we want the area and perimeter to be equal, so…

(27) Lee: Oh, I see. We need to find rectangles with sides lengths a and b that make the area ab equal to the perimeter, which is… *[pauses to write* $ab = 2(a+b)$*]* ...twice the sum of a and b. Now we can solve for a. I'll multiply first. *[mumbles about the distributive property while writing the following]*

$$ab = 2(a+b)$$
$$ab = 2a + 2b$$

Then I can subtract $2a$ from both sides. *[writes the following]*

$$ab - 2a = 2b$$

(28) Matei: Right. And ab minus $2a$ is the same as a times the quantity b minus 2. *[writes the following]*

$$a(b-2) = 2b$$

Then you can divide both sides by b minus 2 and you have a alone on one side. *[writes the following]*

$$a(b-2) = 2b$$
$$a = \frac{2b}{b-2}$$

Oh, so there are *lots* of rectangles!! We can plug in whatever side length we choose for b and we'll always get the length of side a.

(29) Lee: Let's try the two we've found so far. If you put 6 in for b you get $\frac{12}{4} = 3$, and that was our other side length. And if you put 4 in for b you get $\frac{8}{2} = 4$ as our other side length. Yup, it checks!

(30) Chris: So how many rectangles are there that work? Can we really put *any* number in for b and it will give us the a that works for it?

(31) Lee: *[after a brief pause]* Sure. Like if you put in 10 as side length b, that gives us $\frac{20}{8}$ for a. So for a 2.5×10 rectangle, the area is 25 square units and the perimeter is 25 units.

(32) Matei: Guess they are not all going to be your "regular" numbers are they?

(33) Chris: Yeah, but not every number will work… What if you had a side length of 1?

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?
2. Where/how do the students in the dialogue capitalize on their repeated reasoning (MP 8)?
3. What if students don't quickly think of writing an equation? As a teacher, what might you listen for that might help you to decide what (if anything) to suggest to groups as they work on the task?
4. Matei conjectures that the perimeter cannot be odd (line 22). Is this statement true? How does this thinking demonstrate MP 7 (using structure)?
5. How does Chris's question in line 33 of the dialogue provide an opportunity for contextualizing (MP 2)?
6. What student ideas in this dialogue might you follow up on and how would you follow up on them?
7. How might Chris's idea of a "race" (line 8) play out differently in the case of non-square rectangles?
8. In the discussion of squares, it appears that as perimeter increases, so does the area. Is this true for non-square rectangles as well?
9. In line 33, Chris says "What if you had a side length of 1?" Why is this a problem? What other values for b would cause problems?
10. What other constraints might you add to this problem to make the problem more (or less) challenging?

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Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
1 Make sense of problems and persevere in solving them.	This aspect of mathematical practice is evident in most problem solving. In this instance, Chris helps the group to make sense of the problem by offering a starting point to the problem (line 2). Then the group adjusts their path when they realize that they need to go beyond squares to other rectangles. Matei (line 22) suggests that the product of the side lengths needs to be an even number, thus making sense of the numbers that might be solutions (if the numbers are integers). When the students do create an algebraic expression, they use the examples they have already found to verify its accuracy. And they persevere in finding alternative possibilities. The fact that the students create a table and an algebraic expression for the sake of "making sense" shows how interconnected the practice of mathematics is.
2 Reason abstractly and quantitatively.	This practice is about creating a symbolic representation of the problem, manipulating these symbols, and then re-connecting the result to the context of the problem. This is exactly what the students in this problem are doing. More importantly in this practice, though, is the *contextualizing* of the results. Chris's recognition in line 33 that not all numbers that fit the calculations will fit the context is a clear example of MP 2. The geometric problem sets the context and limits the acceptable solutions in the algebra problem.
7 Look for and make use of structure.	Matei's recognition (line 22) that doubling an integer-valued sum will never result in an odd number is an example of attention to structure (of even numbers). Though nothing about this problem restricts the solution to integers, Matei's reasoning about even numbers appears to help the students arrive at one of their solutions.
8 Look for and express regularity in repeated reasoning.	In line 26, Matei notices a pattern and asks "What are we doing each time we try an example?" drawing attention to the students' repeated reasoning. Matei finds regularity that is expressed as equations by substituting variables a and b for the two side lengths and submitting them to the same process. Lee expresses the second part of their repeated reasoning—testing to see if the area and perimeter are equal for some a, b pair they've picked—by creating the equation $ab = 2(a+b)$.

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Commentary on the Mathematics

This task is neither the straightforward "find the area of a rectangle with length l and width w" nor the common "backward" version, "given the area and the length (or width), find the width (or length)." Such questions have unique right answers; there's little call to "think beyond the answer." By contrast, problems with infinitely many correct (and incorrect) answers require genuine engagement in mathematical practice.

The use of letters to stand for numbers appears as early as grade 3 in the *Common Core* content standards (3.OA.D.8). Moreover, the context of the problem—area and perimeter of rectangles—is also addressed in elementary grades. So, the mathematical task in this problem is potentially appropriate much earlier than Algebra 1. But the generalization that these students undertake and their creation of the equation $ab = 2(a+b)$ (content standard A.CED.A.2) and manipulation of that to reveal that $a = \frac{2b}{b-2}$ (content standard A.CED.A.4) is squarely within the High School Algebra content standards.

This dialogue highlights the idea of non-unique solutions: In this case, the fact that infinitely many rectangles satisfy the conditions. The students recognize this early and persevere in finding more than one solution. In fact, they produce a way to identify ALL of the possible solutions to the problem. Because the problem has infinitely many solutions, it can be extended by asking what additional constraints might limit the solutions to a finite number. Can it be limited to exactly 10 solutions? Two? One?

There is a typical misconception potentially at play here—that area and perimeter are inextricably linked. Students often (correctly) understand that a given pair of side lengths produces a unique area and perimeter. However, they are not often asked to think about the idea that a unique perimeter does not fix the area of a rectangle or vice versa. This idea is easily explored through problems similar to the one in this dialogue: If the perimeter of a rectangle is fixed, what's the largest area that can be enclosed? It can also be explored by fixing the area and minimizing or maximizing the perimeter. More on this discussion, focused on triangles, is explored in the Illustration *Finding Triangle Vertices*.

One final note on this dialogue follows an idea that Matei and Chris explore in lines 7 and 8—that once the area and perimeter of a square meet at 4, the perimeter will never again catch up to the area. To further explore this idea, students might graph the two equations $y_1 = x^2$ and $y_2 = 4x$ on the same grid (thus encouraging MP 5 by exploring the graph as a problem-solving tool). This would provide a visual image of the "race" for students and can promote a discussion about the slope/rate of a graph or sequence.

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In line 33 of the student dialogue, Chris asks, "What if you had a side length of 1?" Can you make a rectangle whose perimeter and area are equal if one of the side lengths is 1?

2. For area and perimeter to be numerically equal, sides a and b of the rectangle must be related in this way: $a = \frac{2b}{b-2}$.
   A. What are the limitations for a and b *in the equation*?
   B. Do all the values that work in the equation also solve the original task? Why or why not?

3. In line 22, Matei says that the area can't be odd if we want the perimeter and area of the rectangles to be equal. Matei's reasoning is based on the way perimeter is calculated: adding the side lengths and multiplying by 2 looks like it should give an even number. But in line 31, Lee provides an example of an odd-length perimeter. Under what conditions is Matei's statement true?

4. These students decide to solve their equation for a. What would the equation look like if they had decided to solve for b?

Related Mathematics Tasks

1. Find all the pairs of positive unit fractions whose sum is $\frac{1}{2}$. How does this problem compare with the rectangle problem the students are working on in the dialogue?

2. Are there any triangles whose perimeter and area share the same numeric value? Circles? Pentagons? Hexagons? Is there an analogous equation for any *n*-gon?

3. Are there any rectangular prisms whose surface area and volume share the same numeric value? Spheres? Triangular prisms? Other 3-D shapes?

4. Find all the groups of three positive unit fractions (the three fractions don't all have to be different) that add to $\frac{1}{2}$.

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. Where/how do the students in the dialogue capitalize on their repeated reasoning (MP 8)?

   The students try a few examples with numbers they can easily calculate. After several calculations (line 25), the students want a way to see if they have found all the possible rectangles, and Matei decides to use variables a and b to represent the two lengths. The students put these variables through the same process that they used for testing the numerical values; the result is their equation. The fact that they calculated area and perimeter for many specific examples helped the students express the calculations abstractly, using variables to get their equation.

3. What if students don't quickly think of writing an equation? As a teacher, what might you listen for that might help you to decide what (if anything) to suggest to groups as they work on the task?

   The students in this dialogue, as is quite common, didn't dive right into an equation. Even so, a teacher listening to their conversation might have been reassured about their mathematical thinking at several points. First, in line 5, we see that Lee is very systematic about examining the case of area and perimeter of all possible squares. Even though Lee didn't set up an equation, there is evidence that the students understood the relationship among the calculations involved. A teacher might also have been reassured by the way the students described their calculations for the area and perimeter of the rectangles. Several times Lee (in lines 18, 20, and 24) describes the calculation of perimeter by adding the lengths of adjacent sides and doubling the result. Matei also expresses that idea in line 22 in a more general way. Teachers who hear their students describe calculations this way might reasonably expect students to be ready to use that same reasoning to describe the calculations with variables. Finally, another piece of evidence a teacher might listen for shows up in line 14, when Matei considers non-integer values. Even though Chris (in line 15) doesn't let that line of reasoning progress further, a teacher might infer that the students understand that a more general expression for area and perimeter can capture situations that are otherwise hard to predict (such as rectangles with non-integer lengths).

Clearly, discussions in any classroom will vary, group by group, and these are only examples of the *kinds* of evidence that a teacher might see from a group working on this task. In this case, there was no need to interrupt the students because the way they were working in the beginning—thinking about concrete cases but asking what else there might be—stood a very good chance of getting them to try equations eventually. Patience might be expected to pay off. If students had not been able to keep on track, knowing what they were looking for, intervention—perhaps just asking "What part of the question are you working on now?"—might well have been needed.

4. Matei conjectures that the perimeter cannot be odd (line 22). Is this statement true? How does this thinking demonstrate MP 7 (using structure)?

   Matei's argument about parity (odd and even) assumes whole-number side lengths. Though not applicable for more general lengths, it is a clever observation that makes use of the structure of even numbers—an even number is always 2 × (something). At this point, students are considering only whole numbers. When they generalize later, the conclusion that the area must be even is no longer valid.

   As a side note, finding the possible side lengths of a rectangle when the side lengths *are* restricted to whole numbers is also an interesting problem. This problem (and a similar problem with volume and surface area) is explored in the Related Mathematics Tasks.

5. How does Chris's question in line 33 of the dialogue provide an opportunity for contextualizing (MP 2)?

   A large component of MP 2 is de-contextualizing numbers (or symbols) from a problem, doing some calculation(s) with them, and then (re)contextualizing them to understand or state what the results represent. In this problem, for example, students test a rectangle with side lengths of 2 and 3. They compare 2×3 with $2+3$ doubled. The (re)contextualization is that they understand and state that 6 is the area and 10 is the perimeter and since they are not equal, this rectangle isn't a solution to the problem. They de-contextualize again when they write $a = \frac{2b}{b-2}$, no parts of which represent either area or perimeter. Chris already recognizes a trap when posing the challenge in line 33. If $b = 1$ then $a = -2$. This is a valid solution for the equation, but not for the problem. In this context, a is a *length* and so it can't be negative. Another value that b cannot assume is 2!

   Note that the thinking characterized by MP 2 requires students to be flexible and nuanced. In the case of $b = 1$ (and some other values), students must not ignore the property that length is positive. But in order to engage in the problem at all, students must ignore another property of length and area measure: the units differ in dimension and so, though the perimeter and area can "have the same numerical value," perimeter and area can't be "equal." You may need, at times, to make clear that it is *permitted* to look for numerical equality even when unit equality would make no sense. The nature and context of the problem help you decide when.

6. What student ideas in this dialogue might you follow up on and how would you follow up on them?

    Of course, this depends on your specific situation with students. If you follow up on students' initial thinking, the integer-only investigation can be fruitful. (See Related Mathematics Tasks.) Students' metaphor of a "race" can be explored further through functions and graphs and could be a nice lead-in to the concept of inequalities in two variables. The idea of a product or sum being odd or even could be explored more deeply, and students could revisit this once they depart from integer-only values and see if they can find good examples with odd area. This idea could also be expanded to three numbers and be investigated in the context of volume and surface area.

7. How might Chris's idea of a "race" (line 8) play out differently in the case of non-square rectangles?

    With squares, area wins the race between the numeric values of the area and perimeter of squares because of the constraint that the length and the width both increase.

    When one side length is held constant the "race" can behave quite differently. Look, for example, at the results when one side of the rectangle is 2 and the other varies, or when one side is 3 and the other varies. Students may find it interesting to explain why holding one side constant at 2 (i.e., the table of 2×1, 2×2, 2×3, 2×4, etc.) results in a "boring race."

8. In the discussion of squares, it appears that as perimeter increases, so does the area. Is this true for non-square rectangles as well?

    No. For any pair of *similar* figures, the one with the larger perimeter/circumference will have the larger area. Among *similar* figures, when the perimeter increases, all sides increase in length, so the area increases as well. All squares are similar, but not all rectangles are. Rectangles that measure 5×5 and $50 \times \frac{1}{2}$ have the same area (25) but very different perimeters. Rectangles that measure 8×2 and 5×5 have the same perimeter (20) but different areas. An increase in area is connected to an increase in the *minimum* perimeter. The quadrilateral with the smallest perimeter for a given area is a square.

    Students may also find it interesting to consider what happens with *similar* rectangles. For example, consider a family of similar rectangles. For side lengths a and b, $b = ra$ for some constant r. Then the area of these rectangles could be expressed as $ab = ra^2$, and the perimeter could be expressed as $2(a+b) = 2(a+ra) = 2a(1+r)$. Then if the rectangle's area and perimeter are the same, $ra^2 = 2a(1+r)$, so for $a \neq 0$, $ra = 2(1+r)$ and $a = \frac{2(1+r)}{r}$. Therefore, in each family of similar rectangles, there is one solution. In other words, for each constant r (the ratio of the side lengths), there is one combination of side lengths a and b for which the area and perimeter will be the same.

9. In line 33, Chris says "What if you had a side length of 1?" Why is this a problem? What other values for b would cause problems?

   If $b=1$, then $a=-2$, which is not a possible length for a side of a rectangle. These values satisfy the equation but make no sense in this context. Since neither length can be negative, we have to define a domain for b that will yield only positive values for a. If $b<2$, the equation yields a negative result for a, and if $b=2$, the equation calls for division by 0. So, the end result for the context of this problem is that both side lengths must be greater than 2.

   Another way to investigate this question is to consider what happens when one side has length 2. If side a has length 2, then the area would be $2b$ and the perimeter would be $2(2+b)=4+2b$. But $2b=4+2b$ has no solution; the two sides are never equal, no matter what b is. If you are using the task with students and they have examined the table of areas and perimeters for when one side length is 2 (see Teacher Reflection Question 7), you might ask them to connect what they observed in the table with what this equation says.

10. What other constraints might you add to this problem to make the problem more (or less) challenging?

    Here are three of many options. (1) For students who are having a difficult time starting, you might suggest fixing one of the side lengths, but once they have solved this (one-solution) problem, suggest that they go back to the original and see if this simpler problem gives them any useful ideas. (2) Limiting the side lengths to integer values limits the number of possible rectangles but increases the complexity of the problem because it has to be proven that all possibilities have been found. (3) The numerical value of the area could be fixed at a particular number. This would limit the solutions to one rectangle but would complicate the calculation and ultimately require manipulating a quadratic equation.

Possible Responses to Student Discussion Questions

1. In line 33 of the student dialogue, Chris asks, "What if you had a side length of 1?" Can you make a rectangle whose perimeter and area are equal if one of the side lengths is 1?

   If one of the side lengths is 1, plugging it into the equation the students created (line 28) would give a value of –2 for the other length. While this does work for the equation, you can't have a rectangle with a side length of –2.

2. For area and perimeter to be numerically equal, sides a and b of the rectangle must be related in this way: $a = \frac{2b}{b-2}$.
   A. What are the limitations for a and b *in the equation*?
   B. Do all the values that work in the equation also solve the original task? Why or why not?

      A. This equation works for all real numbers except $b = 2$ since that will make the denominator equal to zero and division by zero is undefined.
      B. In the case $0 < b < 2$, $a < 0$ and side lengths of rectangles can't be negative. In the case $b = 2$, the rectangle does not exist since it has side lengths of 0. And finally $b \le 0$ doesn't work as that would mean one of the sides of the rectangle is 0 or negative. So, to make sense in this context, we must have $b > 2$.

3. In line 22, Matei says that the area can't be odd if we want the perimeter and area of the rectangles to be equal. Matei's reasoning is based on the way perimeter is calculated: adding the side lengths and multiplying by 2 looks like it should give an even number. But in line 31, Lee provides an example of an odd-length perimeter. Under what conditions is Matei's statement true?

   When Matei makes this claim, the students are only considering whole numbers. Under those conditions, Matei's conjecture is true. If any whole number is doubled, the result is even. However, by the end of this dialogue, the students are no longer restricting themselves just to whole numbers. In this case, the area can be odd (as in Lee's example) or a non-integer. Matei's conjecture holds true only when the domain is restricted to integers.

4. These students decide to solve their equation for a. What would the equation look like if they had decided to solve for b?

   If the students solved the equation for b, it would look very similar, except that the "a"s would be in the place of the "b"s and vice versa.

$$ab = 2(a+b)$$
$$ab = 2a + 2b$$
$$ab - 2b = 2a$$
$$b(a-2) = 2a$$
$$b = \frac{2a}{a-2}$$

   This is a solid demonstration of the idea that in any rectangle it is not important which dimension is labeled as the width and which is called the length. They are interchangeable when calculating area and perimeter.

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Possible Responses to Related Mathematics Tasks

1. Find all the pairs of positive unit fractions whose sum is $\frac{1}{2}$. How does this problem compare with the rectangle problem the students are working on in the dialogue?

   It turns out that the number pairs are largely the same and the equation we use is the same.

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2}$$
$$ab\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{2}ab$$
$$a+b=\frac{1}{2}ab$$
$$2(a+b)=ab$$

   This final equation should look very familiar. It turns out that the task of finding all the pairs of positive unit fractions whose sum is $\frac{1}{2}$ is another way of finding the lengths of a rectangle whose area and perimeter are equal, with the additional restriction that the lengths are integers. Standardization doesn't allow for a and b to be anything but integers, so we are much more limited in the possible solutions here. It turns out there are only two possible pairs of unit fractions that sum to $\frac{1}{2}=\frac{1}{4}+\frac{1}{4}$ and $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$. Have students explain why it makes sense that those are the only two possibilities. For example, students might see that $\frac{1}{4}$ is half of $\frac{1}{2}$, so any other pair of fractions that sum to $\frac{1}{2}$ would have to be made up of one fraction that is larger than $\frac{1}{4}$ and one fraction that is smaller than $\frac{1}{4}$. We see that $\frac{1}{3}+\frac{1}{6}$ works, and immediately see that there are no other possible pairs because $\frac{1}{3}$ is the only *unit* fraction that is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$.

   There is also a more algebraic approach:

$$ab=2(a+b)$$
$$ab=2a+2b$$
$$ab-2a-2b=0$$
$$ab-2a-2b+4=0+4$$
$$(a-2)(b-2)=4$$

Since a and b must both be positive integers, there are only three possibilities (using the positive factors of 4):

$$a-2=1 \text{ and } b-2=4\text{, giving } a=3 \text{ and } b=6$$
$$a-2=2 \text{ and } b-2=2\text{, giving } a=4 \text{ and } b=4$$
$$a-2=4 \text{ and } b-2=1\text{, giving } a=6 \text{ and } b=3$$

2. Are there any triangles whose perimeter and area share the same numeric value? Circles? Pentagons? Hexagons? Is there an analogous equation for any n-gon?

Consider a particular convex figure with a fixed area A and perimeter P. Scale the figure by a positive value r. The area of the new figure will be r^2A and the perimeter will be rP. If area and perimeter are the same, $r^2A = rP$, so $r = \frac{P}{A}$. So any figure can be scaled by a factor of $\frac{P}{A}$ to produce a figure whose perimeter and area share the same numeric value.

3. Are there any rectangular prisms whose surface area and volume share the same numeric value? Spheres? Triangular prisms? Other 3-D shapes?

In three dimensions, consider a convex object with volume V and surface area A. This figure, when scaled by a positive value r, will scale to a volume of r^3V and surface area r^2A. Volume and surface area will be equal when $r = \frac{A}{V}$.

4. Find all the groups of three positive unit fractions (the three fractions don't all have to be different) that add to $\frac{1}{2}$.

This problem provides an extension to Related Mathematical Task 1. In particular, note that this problem is related to the problem of finding the sides of a rectangular prism whose volume and surface area are the same numerical value.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$$
$$\frac{2bc+2ac+2ab}{2abc}=\frac{abc}{2abc}$$
$$2bc+2ac+2ab=abc$$

There are 10 groups of three positive unit fractions that sum to $\frac{1}{2}$. This problem extends Related Mathematical Task 1 because we have already found that $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ and

$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$. To find a group of three unit fractions that sum to $\frac{1}{2}$, we can simplify the problem to finding groups of two unit fractions that sum to $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{6}$.

Finding a strategy for finding these groups of unit fractions is an interesting mathematical task for students.

The 10 groups of three positive unit fractions that sum to $\frac{1}{2}$ are:

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}$$
$$\frac{1}{4}+\frac{1}{5}+\frac{1}{20}$$
$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$$
$$\frac{1}{6}+\frac{1}{4}+\frac{1}{12}$$
$$\frac{1}{3}+\frac{1}{12}+\frac{1}{12}$$
$$\frac{1}{3}+\frac{1}{7}+\frac{1}{42}$$
$$\frac{1}{3}+\frac{1}{8}+\frac{1}{24}$$
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{18}$$
$$\frac{1}{3}+\frac{1}{10}+\frac{1}{15}$$

One possible strategy for finding, for example, two fractions that sum to $\frac{1}{3}$ follows from the strategy students in the dialogue used to write the equation relating the two denominators and solving for one of the variables. For example, if $\frac{1}{a}+\frac{1}{b}=\frac{1}{3}$, then $3b+3a=ab$, so $a=\frac{3b}{b-3}$. To find the unit fractions that sum to $\frac{1}{3}$, substitute possible values for b. Fortunately, you only need to try $b=4$, $b=5$, and $b=6$. The value of b can't be less than 4 because that would either require division by 0 (if $b=3$) or result in $a<0$ (if $b=1$ or $b=2$) in the equation $a=\frac{3b}{b-3}$. And the value of b can't be more than 6 because $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$, so for any other cases that will work, one of the fractions must be greater than $\frac{1}{6}$, which means that the denominator for that larger fraction must be less than 6 (and we're arbitrarily choosing that larger fraction to be $\frac{1}{b}$). Out of the three possibilities, only $b = 4$ and $b = 6$ give integer values for *a*, so there are only two pairs of unit fractions that sum to $\frac{1}{3}$: $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$ and $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. A similar strategy can be used to find pairs of unit fractions that sum to $\frac{1}{4}$ and $\frac{1}{6}$.