Similar Triangles

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Similar Triangles* Illustration: This Illustration's student dialogue shows the conversation among three students who are given two similar triangles and asked to find a missing side length. Students use patty paper and their understanding of transformations to help them find the corresponding sides, while in the process making a conjecture about parallel sides in nested triangles.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 5: Use appropriate tools strategically.
MP 6: Attend to precision.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 8–10

Target Content Domain: Geometry,
Similarity, Right Triangles, and Trigonometry (Geometry Conceptual Category)

Highlighted Standard(s) for Mathematical Content

8.G.A.4	Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
G-SRT.A.2	Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
G-SRT.B.5	Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Math Topic Keywords: similar triangles, similarity, transformations, rotations, reflections

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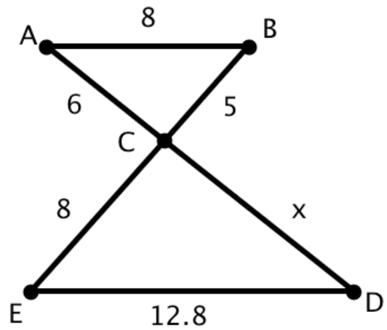
Similar Triangles

Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

In the figure below, the two triangles are similar and points A, C, and D are collinear and points E, C, and B are collinear. What is the length of side CD?



Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students in this dialogue have experience determining congruence of figures through translations, rotations, and reflections. They know that proportional relationships exist between corresponding sides of similar figures, and students are using geometric transformations to help them find the corresponding sides of similar figures in order to use their proportionality.

(1) Lee: I know how to do this. They told us the two triangles are similar so we know the corresponding sides need to be proportional.

(2) Chris: So, that means we need to figure out the scale factor, right?

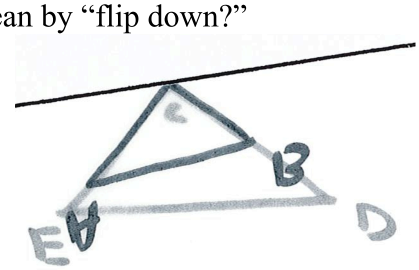
(3) Lee: Sure, but first we need to know which sides are corresponding.

(4) Matei: Ok. Let's figure out what corresponds to what then.

(5) Chris: Will the corresponding sides be lined up if we flip the top triangle down?

(6) Matei: Do you mean reflecting it over C? I mean, over the horizontal line through C? I don't think that works. I think the corresponding sides won't line up.

(7) Lee: Let's try it. *[Lee traces the entire figure onto a piece of tracing paper, and folds the paper so that the vertical angles, $\angle ACB$ and $\angle ECD$, coincide.]* Wait, so is this what you mean by "flip down?"



(8) Chris: Yeah, that's what I meant. You reflected the triangle. I guess it's not over the horizontal line through C, but it's a reflection and…

(9) Matei: Okay, so see… The sides that are now lined up with each other are not *corresponding* sides.

(10) Chris: How do you know?

(11) Matei: Well, for one thing, these bottom lines, AB and ED, are not parallel and I think they should be.

(12) Chris: Is that enough to know the overlapping sides aren't corresponding, though?

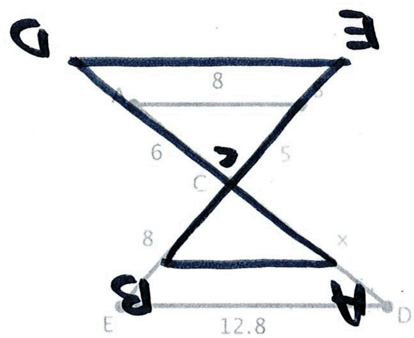
(13) Lee: We could check to see if their side lengths are proportional. The question is whether the ratios of EC to AC and ED to AB are the same. *[writes the following]*

$$\frac{8}{6} = 1.\overline{3} \qquad \frac{12.8}{8} = 1.6$$

These pairs of sides aren't proportional so they can't be corresponding!

(14) Chris: What if we rotated the diagram instead of reflecting it. Would that help?

(15) Lee: Oh, I like that. I'll try it. *[Lee unfolds the tracing, places it on top of the original diagram, and then rotates it 180 degrees around point C.]*



(16) Matei: Look! This time the bottom lines of the two triangles are parallel.

(17) Lee: Well, we still don't know for sure whether that matters; that's still just a conjecture. But we *can* check to see if the overlapping sides are proportional. *[writes the following]*

$$\frac{12.8}{8} = 1.6 \qquad \frac{8}{5} = 1.6$$

It works! They're proportional.

(18) Matei: So… side BC corresponds with side EC, and side AC corresponds with side DC.

(19) Lee: And the scale factor is 1.6.

(20) Chris: So, what's the length of side CD?

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. Initially, students try folding to see whether a reflection would superimpose corresponding sides. They decide—first by appearance and then by checking scale factors—that reflection does not work in this case, but that rotation will. Is there a case in which reflection *would* superimpose corresponding sides, but rotation would *not*? If so, give an example; if not, show why not.

3. What challenges and/or misconceptions might students have when working with similar triangles?

4. The dialogue ends with Chris reminding the group of the original, still unanswered question. What MP is Chris meeting here? If students in your class had reasoned their way through the problem as the students in the dialogue do and yet were unable to answer the question that Chris recalls, how might you help?

5. How do the tools the students use allow them to continue reasoning about the problem or to engage in MPs? Are there additional tools you would recommend for students when working on this task?

6. In lines 11 and 16, Matei claims that if the sides that "line up" are corresponding sides, then the third sides must be parallel. In line 12, Chris questions whether the converse is true: If the third sides are parallel, does that guarantee that the superimposed sides correspond?

7. The students in the dialogue calculate the scale factors and test for proportionality, determining *arithmetically* which sides correspond. What information could be given in the task to let students determine *geometrically* (that is, without calculating the scale factors) which sides are corresponding?

8. How might you modify this mathematics task to make it more accessible or more challenging and thus suit a wide variety of students?

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Make sense of problems and persevere in solving them.	In lines 1–4 of the dialogue, we see students making sense of the geometric information they have been given (that the two triangles are similar) and the implication of that information (that corresponding sides will be proportional). Another important component of MP 1 is persevering in solving problems. In this dialogue, students try different transformations of $\triangle ABC$ to find the corresponding sides. After each transformation, they check to see if the sides that seem to "line up" are in fact corresponding (i.e., proportional). We also see students conjecturing about the possible meaning of having a set of parallel sides in the transformed nested triangles (lines 11–12).
 Use appropriate tools strategically.	Using tracing paper to test their ideas is a strategic choice. Tracing paper is a simple low-tech tool to help organize students' search and help them test out both reflections and rotations. They also use it to show each other what they mean, adding precision to their communication as they try to articulate ideas about which sides should be compared, and aiding understanding for all three students.
 Attend to precision.	Imagine how this dialogue might have been different if the students were using a less precise notion of similarity (e.g., "same shape, different size"). The students are able to continue refining their ideas because they have some notions and questions (and share those notions and questions with each other) about the more precise meaning of similarity—they are examining correspondence through geometric transformations, and they are developing their understanding of how similarity involves scaling or dilation by a scale factor. The students use a mixture of informal and formal language as they communicate with each other about their ideas—talking about scaling, flipping, reflecting, and rotating. The students demonstrate attention to precision when they push each other to explain further or demonstrate (with appropriate tools! See MP 5) their understanding of these terms and to articulate more clearly their ideas. For example, in casual speech, you can't be sure whether "turn the letter R upside down" means ᴚ or ꓤ, and the same is true of "flip the letter R upside down." When Chris talks about "flipping" in line 5, Matei questions Chris about what it means to flip $\triangle ABC$ and Lee uses the tracing paper to check that they have the same definition of "flipping." Chris then clarifies with more precise language in line 8, "Yeah, that's what I meant. You reflected the triangle."

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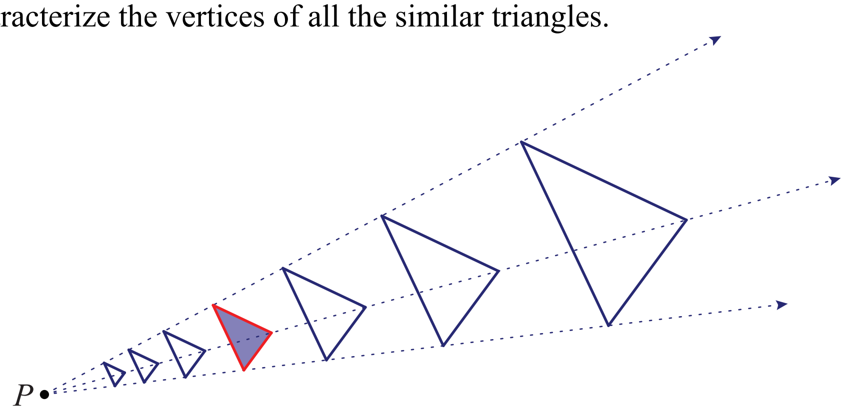
Look for and make use of structure.	Students are using structure in their implicit understanding that a series of transformations can be used to change $\triangle ABC$ into $\triangle DEC$. Even though students do not perform the final dilation, searching for the appropriate series of transformations is what they are actually doing when looking for the corresponding sides of the similar triangles. Also students are looking for structure when they conjecture about the implications of a parallel vs. nonparallel third set of sides in the nested triangles (lines 11–12, 16).

Commentary on the Mathematics

Evidence of the Content Standards – Understanding similarity as a series of geometric transformations (8.G.A.4, G-SRT.A.2, and G-SRT.B.5)

Understanding congruence in terms of geometric transformations (i.e., that congruent figures are ones that can be superimposed on each other through a set of reflections, translations, and/or rotations—or "rigid" transformations) is a first step toward understanding geometric similarity. The next step is to consider the role of dilation: Two figures are similar if there exists a dilation of one that can be established as congruent to the other (again, by rigid transformations). In this problem, the students use the rigid transformations to help them determine which sides are corresponding so they can be compared through dilation. While the students in this dialogue do not explicitly talk about dilation, they are using the properties of dilation when they refer to scaling or the proportionality of figures to test corresponding sides. Just as rotations about a point and translations with respect to a point generate infinitely many images of an object, infinitely many similar figures (or dilated images) can be produced through dilation about a given point, from tiny to large. All along the dotted lines below—rays originating at the center of dilation (point P) and passing through the vertices of the shaded triangle—are the vertices of triangles similar to the shaded one. Conceptualizing infinite sets like this is the basis of a kind of abstract reasoning required in geometry. Of course, vertices of *non-similar* triangles can also be set on those rays (try it!), but their distances from the center of dilation then don't fit the proportional pattern that characterize the vertices of all the similar triangles.



Shrinking down to a point—an image of change as a continuous process—can be extended to show a triangle "coming out the other side" of that point (extending the rays into lines). That is also a dilation by a negative scale factor. If the original triangle "comes out the other side" just far enough to achieve its original size, it appears to have been rotated 180 degrees. Relationships among the transformations are interesting to explore—and not all are continuous. Reflection on the plane, for example, has a discrete nature: While one can rotate more or less, or dilate more or less, reflection in the plane is all or nothing.

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. The dialogue ends with Chris asking, "What's the length of side CD?" What is it? Explain your reasoning.

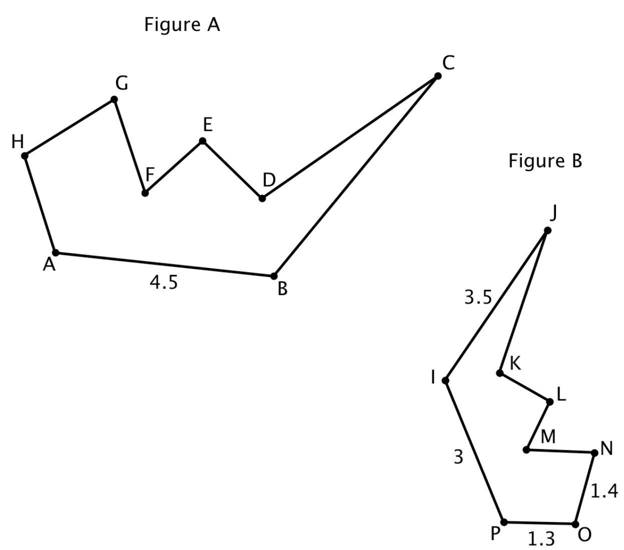
2. How do students use reflections and rotations in this dialogue? What other transformation would students need to perform at the end of the dialogue to turn $\triangle ABC$ into $\triangle DEC$?

3. In lines 11 and 16, Matei claims that if the sides that "line up" are corresponding sides, then the third sides must be parallel. In line 12, Chris questions whether the converse is true: If the third sides are parallel, does that guarantee that the superimposed sides correspond? Does it?

4. If you start with a square, can you create a similar square by increasing the length of each side by the same amount (e.g., adding two inches to each side)? What if you start with a non-square rectangle? Will increasing the length of each side by the same amount result in a similar rectangle? In each case, explain why or why not.

Related Mathematics Tasks

1. Figures A and B are similar. What is the length of $\overline{HG}$? Explain your reasoning.



Similar Triangles

2. How can you check if two circles are similar? How can you check if two circles are congruent?

3. Two similar triangles have areas of 4 square inches and 36 square inches. If the length of one side of the smaller triangle is 1.5 inches, what is the length of the corresponding side of the larger triangle?

4. Jose is wondering how far apart two docks are on the other side of a river. He knows that the river is 300 yards across at all points in this section of the river. He has a measuring tape so he can measure distances on his side of the river. How can he use similar triangles to find out the distance between the two docks?

Similar Triangles

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

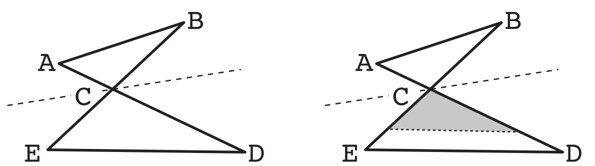
Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

    Refer to the Mathematical Overview for notes related to this question.

2. Initially, students try folding to see whether a reflection would superimpose corresponding sides. They decide—first by appearance and then by checking scale factors—that reflection does not work in this case, but that rotation will. Is there a case in which reflection *would* superimpose corresponding sides, but rotation would *not*? If so, give an example; if not, show why not.

    Here is an example of similar triangles joined at congruent angles formed by two intersecting lines, in which a reflection (over the dotted line) does superimpose corresponding sides. The shaded triangle in the second picture shows the image of $\triangle ABC$ after it has been reflected over the dotted line. As Matei had conjectured (line 11), if the superimposed sides are corresponding, then the third corresponding sides must be parallel.



    The image of $\overline{AB}$ illustrates the "side splitter" theorem: $\overline{CE}$ and $\overline{CD}$ are split proportionally by the reflection of $\overline{AB}$.

3. What challenges and/or misconceptions might students have when working with similar triangles?

    Similarity can be a challenging concept for students to explore. One difficulty highlighted in the dialogue is the problem of identifying corresponding sides. Students may also be unsure how to set up a proportion, or may think first of additive instead of multiplicative growth. Students may also stumble if they calculate the scale factor "backwards" by dividing the length of a pre-image side by the length of the corresponding image side (the reciprocal of the scale factor). To deal with that, students might be encouraged to pause and check the reasonableness of their calculated scaling factor (is it <1 or >1).

4. The dialogue ends with Chris reminding the group of the original, still unanswered question. What MP is Chris meeting here? If students in your class had reasoned their way through the problem as the students in the dialogue do and yet were unable to answer the question that Chris recalls, how might you help?

   Monitoring where we are in a problem-solving process and evaluating progress is one essential element of MP 1. The students in the dialogue work through the meaning and reasoning, and then remind themselves of the original problem and see how their work can be applied to help them solve it. It may seem hard to imagine students having known *what* reasoning to use and then not knowing how to use it, but this does occasionally happen. One strategy for helping students over this hurdle is just to ask them a "naïve" question like, "So, just to catch me up, what have you figured out so far?" which helps them reiterate what they have worked out and *why* they chose that route. Or you might be more leading: "What is changing between the two triangles?" and, if necessary, "Are the different parts changing at the same rate?" You might also ask them to stretch their thinking with a question like, "What is an estimate of what the side lengths would be of a similar triangle that is in between the size of these two triangles?"

5. How do the tools the students use allow them to continue reasoning about the problem or to engage in MPs? Are there additional tools you would recommend for students when working on this task?

   Refer to the Mathematical Overview for information about the use of tools as characterized by MP 5: Use appropriate tools strategically. Tracing paper is a useful tool for helping students visualize and explore geometric transformations. Cut-outs and, in some circumstances, graph paper reproductions can also help students visualize. Geometry software allows experimentation without requiring precise cutting or paper folding. With software, students can also measure side lengths, notice whether an increase in side-lengths progresses clockwise or counter-clockwise, and think about whether a reflection of one of the triangles is needed to compare corresponding sides of the two triangles. The ratios of the shortest sides, middle sides, and longest sides can then be compared to see if they are the same. Geometry software also allows students to use geometric transformations to compare the figures in ways like those in the dialogue.

6. In lines 11 and 16, Matei claims that if the sides that "line up" are corresponding sides, then the third sides must be parallel. In line 12, Chris questions whether the converse is true: If the third sides are parallel, does that guarantee that the superimposed sides correspond?

   Yes. In the case of nested triangles, having a third set of sides that are parallel guarantees that the two triangles are similar and that the sides that "line up" are corresponding and, therefore, proportional. This is related to the theorem that states that if a line segment with endpoints on a triangle is parallel to a side of the triangle, then the line segment splits the two intersecting sides proportionally and the line segment is also proportional to the parallel triangle side. (See CCSS High School Math Content Standard G-SRT.B.4.)

7. The students in the dialogue calculate the scale factors and test for proportionality, determining *arithmetically* which sides correspond. What information could be given in the task to let students determine *geometrically* (that is, without calculating the scale factors) which sides are corresponding?

    The task could state that $\angle BAC \cong \angle EDC$. This would then allow students to make use of the congruence of vertical angles and the 180 degree angle sum to determine that $\angle ABC \cong \angle DEC$, all of which help sort out the *order* of the corresponding parts of the similar triangles. Or the task could state that $\overline{AB}$ and $\overline{DE}$ are parallel, in which case the alternate interior angles are congruent. In either case, once one knows which angles are congruent, the sides opposite them are the corresponding sides.

8. How might you modify this mathematics task to make it more accessible or more challenging and thus suit a wide variety of students?

    Geometry software for students to use can allow them to explore the ratios of side lengths in a figure like this efficiently, and also explore how geometric transformations might relate to congruence and similarity. For students who need a suggestion of where to start, you might explicitly ask them what they would have to do in order to construct the figure of the specified shape, and assure that no matter how the points of one triangle are dragged around, distorting the shape of that triangle, the second triangle *remains* similar to the first. (Construct line j through points E and B, and line k through points A and D, intersecting at point C. Construct $\overline{AB}$, creating a single triangle. From point E, construct a *line parallel* to $\overline{AB}$—a key property of this diagram even hinted at in the students' dialogue—intersecting line k at point D. Construct $\overline{ED}$, $\overline{AD}$, and $\overline{EB}$ and hide the full lines. Dragging points A, B, or E will preserve the similarity unless a point dragged across C eliminates C as an intersection.) A real-world context (e.g., using similar triangles to measure unknown heights or distances) can enrich the problem for some students.

    For an extension, you could have students consider what "similar triangles" might mean on a sphere. They might start by imagine drawing *one* triangle on the parking lot, and then imagine such a thing filling a parking lot the size of the Pacific Ocean. We have to extend the meaning of "straight line" to mean "shortest path" and "not veering to one side of the other," both according with what we *want* "straight" to mean. Then, one "straight line" can be the equator (Figure 1). A second straight line can depart at right angles to the equator and go through the North and South poles (Figure 2).

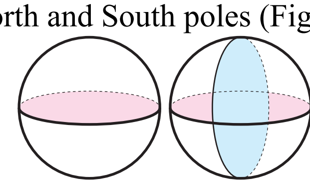


Figure 1 Figure 2

    Now comes the fun. Consider only one segment from equator to North Pole, and then, from a different point on the equator, another segment to the North Pole (Figure 3). The result is a triangle with two 90° angles at the equator and a third angle P such that $0^{\circ} < m(P) < 180^{\circ}$ at the North Pole (Figure 4). The angle sum is, in fact, related to the size of the triangle! (Play with it, first imagining moving one of the equator points closer to or farther from the other, and then by moving the pole intersection closer to the equator. The smaller the triangle, the smaller the angle sum.) Therefore, if you explore a setup like the one in this problem—two triangles that meet at a common vertex so that the adjacent edges lie in a pair of great circles—it will turn out that these triangles can't be similar unless they are congruent. That is, they can't have the same three angles (which would mean the same angle sum) and also be of different size. If they *are* congruent, the transformation that superimposes one on the other could be a rotation of 180° or a reflection.

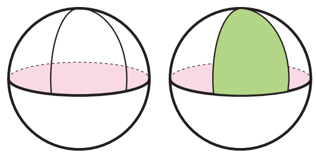


Figure 3 Figure 4

Possible Responses to Student Discussion Questions

1. The dialogue ends with Chris asking, "What's the length of side CD?" What is it? Explain your reasoning.

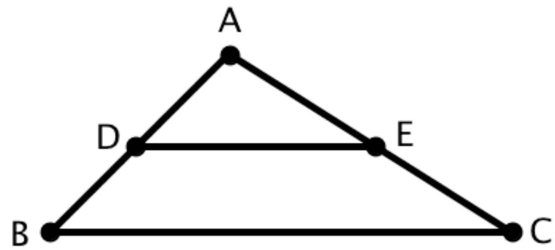
   The length of $\overline{CD}$ is $\frac{48}{5}$ (or 9.6). This can be found by multiplying the length of $\overline{AC}$ (the side corresponding to $\overline{CD}$) by $\frac{8}{5}$, which is the scale factor in the dilation from $\triangle ABC$ to $\triangle DEC$.

2. How do students use reflections and rotations in this dialogue? What other transformation would students need to perform at the end of the dialogue to turn $\triangle ABC$ into $\triangle DEC$?

   The students used tracing paper to reflect and rotate one of the triangles to attempt to visually line up pairs of sides before they tested, by calculating ratios, whether the sides that lined up after the transformations were corresponding sides. Once corresponding sides of the triangles line up, one triangle can be stretched out (dilated) to match the other. A dilation of $\triangle ABC$ around point C, with a scale factor of $\frac{8}{5}$, will transform it into $\triangle DEC$.

3. In lines 11 and 16, Matei claims that if the sides that "line up" are corresponding sides, then the third sides must be parallel. In line 12, Chris questions whether the converse is true: If the third sides are parallel, does that guarantee that the superimposed sides correspond? Does it?

   If two triangles are nested so that two pairs of their sides "line up" (are superimposed) and the third pair of sides is parallel, then the sides that line up are corresponding and proportional. This is a consequence of the theorem, sometimes referred to as the "side splitter theorem," that a line parallel to one side of a triangle divides the other two proportionally. (See CCSS High School Math Content Standard G-SRT.B.4.) For example, in the drawing below, if you know $\overline{DE} \parallel \overline{BC}$, then you know that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$, which means that the sides that "line up" are proportional and, therefore, corresponding sides of two similar triangles.



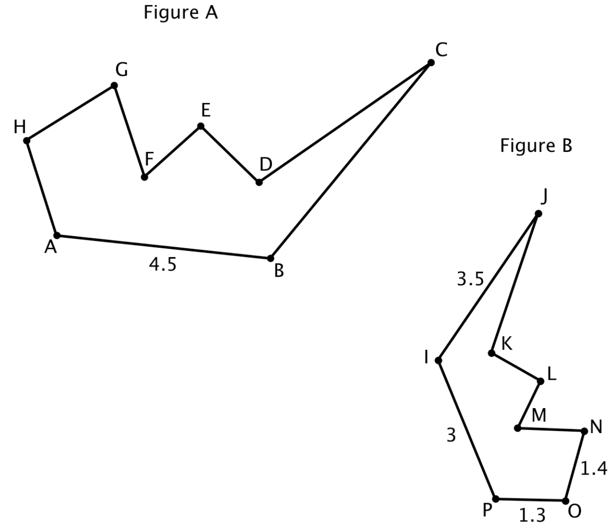
4. If you start with a square, can you create a similar square by increasing the length of each side by the same amount (e.g., adding two inches to each side)? What if you start with a non-square rectangle? Will increasing the length of each side by the same amount result in a similar rectangle? In each case, explain why or why not.

   In the case of squares, all of the sides start out the same length, so adding the same amount to each side keeps all the sides equal and, therefore, proportional to the original square. In the case of a non-square rectangle, adding the same amount to every side does not preserve the ratio of the sides; thus, it does not result in a similar rectangle.

Similar Triangles

Possible Responses to Related Mathematics Tasks

1. Figures *A* and *B* are similar. What is the length of $\overline{HG}$? Explain your reasoning.



Because these figures' features are so distinctive, we can judge corresponding sides just by eye, as long as we are told that they are, indeed, similar. To find the scale factor, we must find a pair of corresponding sides, both of which are measured. $\overline{IP}$ and $\overline{AB}$ are that pair, and they give us a scale factor of 1.5. We can determine that the length of $\overline{HG}$ is 2.1 by multiplying the length of $\overline{NO}$ (corresponding to $\overline{HG}$) by the scale factor.

2. How can you check if two circles are similar? How can you check if two circles are congruent?

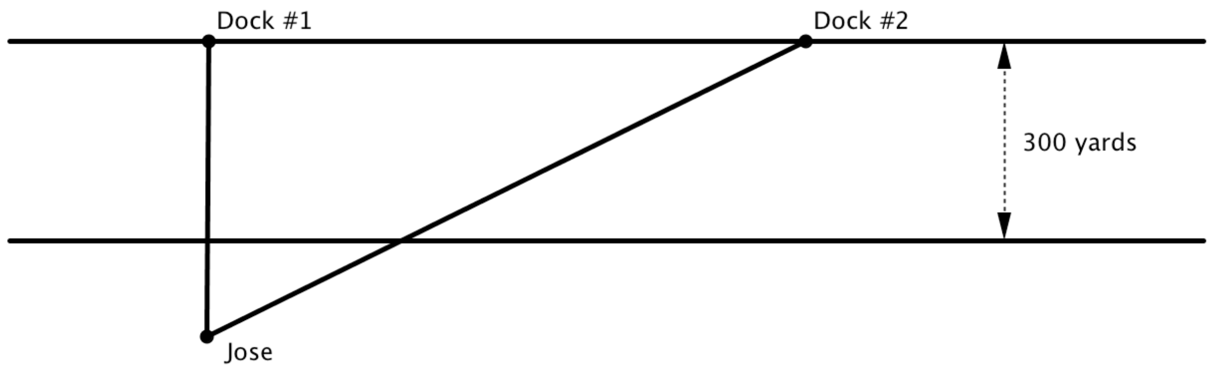
Checking if two circles are similar won't take long—all circles are similar! No matter how much you dilate a circle, all measurements within the circle will stay proportional. You can always translate a dilated image of one circle to match another circle—you don't even need any rotations or reflections. To check if two circles are congruent, you need only to know one measurement (radius, diameter, circumference, or area) of each and compare or compute.

3. Two similar triangles have areas of 4 square inches and 36 square inches. If the length of one side of the smaller triangle is 1.5 inches, what is the length of the corresponding side of the larger triangle?

The length of the corresponding side of the larger triangle is 4.5 inches because the area of the larger triangle is 9 times the area of the smaller triangle, so all linear measurements must be related by a scale factor of 3 (because 3 squared is 9). Three times 1.5 is 4.5.

Similar Triangles

4. Jose is wondering how far apart two docks are on the other side of a river. He knows that the river is 300 yards across at all points in this section of the river. He has a measuring tape so he can measure distances on his side of the river. How can he use similar triangles to find out the distance between the two docks?



Jose could stand back several yards from the shore in a place where one dock is directly across the river from him. He can look at the other dock from this location, and see what point on the shore lies directly between him and the second dock. Two similar triangles have now been formed—one is the triangle with Jose as one vertex and the two docks as the other two vertices, and the other is the triangle with Jose as one vertex, and the points on the shore that are directly between Jose and the two docks. The two triangles are similar because the river banks are parallel. Jose can measure all side lengths of the smaller triangle, and he knows the side length of one side of the larger triangle because it is 300 yards plus the distance from where he is standing to the river's edge. The corresponding side lengths of Jose to the river's edge and Jose to the dock that is directly across the river can be compared with the scale factor for the two similar triangles, and then that scale factor can be used to find the missing side length between the two docks.