Sum of Rational and Irrational Is Irrational

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Sum of Rational and Irrational Is Irrational* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to determine if a sum of a rational and irrational number is rational. Students determine that the sum is in fact irrational and after trying several similar sums, they develop a proof for why the sum of a rational and irrational is always irrational.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 3: Construct viable arguments and critique the reasoning of others.
MP 7: Look for and make use of structure.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 8–9

Target Content Domain: The Real Number System (Number and Quantity Conceptual Category)

Highlighted Standard(s) for Mathematical Content

N-RN.B.3	Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Math Topic Keywords: rational numbers, irrational numbers, proof, proof by contradiction, indirect proof

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Is $\sqrt{2}+\frac{1}{2}$ a rational number?

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Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have previously learned the difference between rational and irrational numbers and have seen examples of each. They are now investigating what sums of different types of numbers will produce.

(1) Chris: What's a rational number again? Isn't it a fraction?

(2) Lee: Yeah, it's a number that can be written as $\frac{a}{b}$ where a and b are integers and b is not equal to zero.

(3) Chris: Ok. So $\sqrt{2}$ is irrational and $\frac{1}{2}$ is rational. What does that tell us about the whole thing, $\sqrt{2}+\frac{1}{2}$?

(4) Matei: Well, we basically want to figure out if the following is true:

$$\sqrt{2}+\frac{1}{2}=\text{rational number}$$

Let's try rewriting it as:

$$\sqrt{2}=\text{rational number}-\frac{1}{2}$$

(5) Chris: Well… wait! That doesn't look right! Isn't the difference of two rational numbers also rational? How can an irrational number equal a rational number?!?!

$$\underbrace{\sqrt{2}}_{\text{irrational number}}=\underbrace{\text{rational number}-\frac{1}{2}}_{\text{rational number}}$$

(6) Lee: You're right—that doesn't make sense! So what does that mean?

(7) Matei: Well, we assumed $\sqrt{2}+\frac{1}{2}=\text{rational number}$. I guess that must be false. The only other option is for it to be irrational. I wonder though…what would have happened if we started with something different?

(8) Chris: Like what?

(9) Matei: Well, let's try $\sqrt{2}-\frac{1}{2}$.

(10) Lee: So let's see if $\sqrt{2}-\frac{1}{2}=\text{rational number}$. We can rewrite that as $\sqrt{2}=\text{rational number}+\frac{1}{2}$.

(11) Chris: But we have the same problem as before. An irrational number can't be equal to a rational number. So that means $\sqrt{2}-\frac{1}{2}$ is irrational, too.

(12) Matei: What if we tried $\sqrt{2}+2$?

(13) Lee: Well, we can say $\sqrt{2}+2=\text{rational number}$ or $\sqrt{2}=\text{rational number}-2$.

(14) Chris: That doesn't work either. $\sqrt{2}+2$ has to be irrational.

(15) Matei: Let's see if $\sqrt{7}+2$ is rational.

(16) Chris: We're just going to end up with $\sqrt{7}=\text{rational number}-2$ so $\sqrt{7}+2$ is irrational.

(17) Lee: Hmm… I'm starting to see a pattern here. Every time we have:

$$\text{irrational number}\pm\text{rational number}=\text{rational number}$$

we end up rewriting that as:

$$\text{irrational number}=\text{rational number}\mp\text{rational number}$$

(18) Chris: You're right! And that ends up giving us something that doesn't make sense. The right-hand side is always a rational number, and that can't be equal to an irrational number.

(19) Matei: Well, I guess that can only mean one thing:

$$\text{irrational number}\pm\text{rational number}\neq\text{rational number}$$

So the sum of an irrational and a rational number must be irrational.

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. The students in this dialogue are able to recognize that the sum of two rational numbers is always rational (line 5 of the dialogue). If students were not able to see this for themselves, how could you help them come to this realization?

3. What challenges might students encounter if they had tried to show that $\sqrt{2}+\frac{1}{2}$ is irrational?

   How would you help students overcome that challenge?

4. If students have trouble understanding what is meant by a contradiction, what other examples of contradictions can you offer that students might be more familiar with?

5. In line 3, Chris says "$\sqrt{2}$ is irrational." Prove that this is true.

6. What is the product of a rational and irrational number? Prove your conjecture.

7. Make a conjecture about the sum of two irrational numbers and support your reasoning.

8. Write down 20 irrational numbers. What do you notice?

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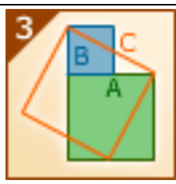
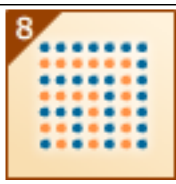
Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Construct viable arguments and critique the reasoning of others.	According to MP 3, "They [students] make conjectures and build a logical progression of statements to explore the truth of their conjectures." In this dialogue, students make the conjecture that $\sqrt{2}+\frac{1}{2}$ is rational (line 4) and explore to see if that conjecture is true. In the process, they find a contradiction (lines 5–7), thereby "distinguish[ing] correct logic or reasoning from that which is flawed." Even though the students do not identify their process as such, they are in fact using the same logic as that found in a proof by contradiction (indirect proof).
 Look for and express regularity in repeated reasoning.	Students in this dialogue "notice if calculations are repeated, and look both for general methods and for shortcuts." They test several numerical cases (lines 7–16) before they make a generalization about the sum of an irrational and rational number (lines 17–19). Working through several concrete examples allows students to see that the contradiction they found when testing to see if $\sqrt{2}+\frac{1}{2}$ is rational was not limited to that case alone. By seeing that the conjecture was false for several examples, they are able to develop a new, more general conjecture—that the sum of an irrational and rational number is irrational—and begin to understand why that is true.
 Look for and make use of structure.	In the dialogue, students "can see complicated things… as single objects." In line 5, Chris is able to see that the expression "$\text{rational number}-\frac{1}{2}$" is a rational number by knowing something about the structure of rational numbers and the way they can be added/subtracted together, even though the student never fully explains the thinking used in the dialogue. Chris's ability to identify expressions as rational is also based on the use of "clear definitions" (MP 6) for a rational number as a fraction of two integers with a non-zero denominator.

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Commentary on the Mathematics

Indirect proofs

Indirect proofs can be challenging at first because of what may appear as backwards logic. Getting a contradiction (a false statement) in the process of trying to prove something and using that contradiction to draw a conclusion may seem bizarre to students. This type of proof is also not always purposeful. Sometimes in the process of trying to prove a statement, you may come across a contradiction that tells you something about the original statement you are trying to prove. Indirect proofs are very beneficial when trying to prove things that are difficult to quantify either due to their inability to be expressed in a generic form, as in the case of irrational numbers, or due to their inability to be measured, as in the case of an infinite number of objects (e.g., prime numbers). In these cases, making an assumption that your number is rational or that prime numbers are finite leads us to a contradiction that helps prove our solution is irrational or our set is infinite.

Evidence of the Content Standards

In the dialogue students need to determine if a sum is a rational number or not. While answering the question they come to a contradiction (line 5) which leads them to develop an argument for why the sum of a rational and irrational number is in fact irrational (lines 17–19) (N-RN.B.3).

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. What is a rational number? What is an irrational number?

2. In line 5 of the student dialogue, Chris identifies rational number $-\frac{1}{2}$ as a rational number. How do you think the student knows that it is a rational number?

3. How do students realize that $\sqrt{2}+\frac{1}{2}$ is an irrational number?

4. How do students come up with the conclusion that the sum of any rational and irrational number is always irrational?

Related Mathematics Tasks

<u>On odd and even integers</u>

1. For any integer n, what expression can be written to represent all even numbers? What expression can be written to represent all odd numbers? Explain how you came up with your expressions.

2. Explain why the sum of an even and an even is always even.

3. Explain why the sum of an odd and an odd is always even.

4. Explain why the sum of an even and an odd is always odd.

5. Explain why the product of an odd and even is always even.

6. Explain why the product of odd and odd is always odd.

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On rational and irrational numbers

7. Find a pair of irrational numbers whose sum is **rational**. Find a pair of irrational numbers whose sum is **irrational**.

8. What can you say about the product of two irrational numbers? Support your answer.

9. Is $\sqrt{2} \cdot \frac{1}{2}$ a rational number? Explain why or why not.

10. What is the product of a rational and irrational number? Prove your conjecture.

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. The students in this dialogue are able to recognize that the sum of two rational numbers is always rational (line 5 of the dialogue). If students were not able to see this for themselves, how could you help them come to this realization?

   One way to help students realize that the sum of two rational numbers is rational would be to have them pick two rational numbers and calculate their sum while delaying the final computation in order to see the structure behind the calculation. For example:

$$\frac{2}{3}+\frac{5}{7}=$$
$$\frac{2\cdot 7}{3\cdot 7}+\frac{5\cdot 3}{7\cdot 3}=$$
$$\frac{2\cdot 7+5\cdot 3}{3\cdot 7}$$

   After trying several examples, students will see that they always get a rational numerator and denominator of the form $\frac{ad+bc}{bd}$. In fact, replacing the integers with variables, students can then see that any two generic rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, will produce a sum of $\frac{ad+bc}{bd}$, which is rational. Having students try numerical examples first, before they prove the general case based on what they have learned from their numerical calculations, is an example of engaging in MP 8.

3. What challenges might students encounter if they had tried to show that $\sqrt{2}+\frac{1}{2}$ is irrational? How would you help students overcome that challenge?

   Had the students tried to prove that $\sqrt{2}+\frac{1}{2}$ is irrational, they would be trying to show that $\sqrt{2} = \text{irrational number} - \frac{1}{2}$ is true by arguing that $\text{irrational number} - \frac{1}{2}$ is irrational. However, students have no way of knowing whether $\text{irrational number} - \frac{1}{2}$ is irrational since irrationals cannot be easily identified from a standard form like $\frac{a}{b}$ as in the case of rational numbers. In fact, proving that a number is irrational must be done using a proof by contradiction: first assume the number is rational, and then see that the assumption leads to a contradiction, which means the assumption is false and the number is irrational.

4. If students have trouble understanding what is meant by a contradiction, what other examples of contradictions can you offer that students might be more familiar with?

   One scenario in which beginning algebra students use contradiction is when checking solutions to equations. For example, when checking to see if 2 is a solution to the equation $2x+5=10$, we get the following:

$$2x+5=10$$
$$2(2)+5=10$$
$$4+5=10$$
$$9=10$$

   Since using 2 as a solution leads to the contradiction that $9=10$, you know that 2 cannot be a solution.

5. In line 3, Chris says "$\sqrt{2}$ is irrational." Prove that this is true.

   Start by seeing what happens if $\sqrt{2}$ is rational. This means:

$$\sqrt{2}=\frac{a}{b}$$
$$2=\frac{a^2}{b^2}$$
$$2b^2=a^2$$

   Next rewrite a as $a=2^m u$ where $m \ge 0$. This means:

$$2b^2=a^2$$
$$2b^2=(2^m u)^2$$
$$2b^2=2^{2m}u^2$$

Since there is an odd number of the factor "2" on the left side of the equation and an even number of the factor "2" on the right side of the equation, we have a contradiction. Because of the contradiction, we know that the assumption that $\sqrt{2}$ is rational must be false and, thus, $\sqrt{2}$ is irrational.

6. What is the product of a rational and irrational number? Prove your conjecture.

   The product of a rational and irrational number is irrational.
   Consider $z = x \cdot y$ where x is irrational and y is rational and not equal to 0. You can rewrite the equation as $\frac{z}{y} = x$. By substituting several rational numbers for z and y, you can see that x would always come out rational and lead to a contradiction. In general, the quotient of two rational numbers (given the denominator is not 0) is rational. Because a rational z contradicts our definition that x is irrational, this must mean z is irrational. Therefore, the product of a rational and irrational number is irrational.

7. Make a conjecture about the sum of two irrational numbers and support your reasoning.

   The sum of two irrational numbers can be either rational or irrational. For example $\sqrt{2} + (-\sqrt{2}) = 0$ is a case in which two irrational numbers add up to give you a rational number. However $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is a case in which two irrational numbers add up to give you an irrational number. (We know $2\sqrt{2}$ is irrational because it is the product of a rational and irrational number.)

8. Write down 20 irrational numbers. What do you notice?

   This is actually a great activity to try in class with students. Ask students to give an example of an irrational number…then another, and another, and then another. For a while, they may give similar examples, e.g., $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. However, they will eventually tire of the "same thing" and try something like π. If you continue, you may hear things like $\sqrt[3]{7}$, $\sqrt[5]{9}$ (in words, any $\sqrt[n]{a} \notin \mathbb{Q}$ unless $a = b^n$), $\sqrt{2} + 1$, $2\sqrt{3}$, $1000 - \sqrt{2}$, etc. The activity exercises what students know while expanding their mental image of the category.

Possible Responses to Student Discussion Questions

1. What is a rational number? What is an irrational number?

   Rational numbers can be written as the quotient of two integers where the denominator is not zero. In other words, they can be written in the form $\frac{a}{b}$ if b is not zero. Irrational numbers cannot be written as the quotient of two integers.

2. In line 5 of the student dialogue, Chris identifies rational number $-\frac{1}{2}$ as a rational number. How do you think the student knows that it is a rational number?

   Chris realizes that when two fractions are subtracted, the result is always another fraction, also known as a rational number. In this example, replace "rational number" with $\frac{a}{b}$, and the whole expression becomes $\frac{a}{b}-\frac{1}{2}$, which is the same as $\frac{2a-1b}{2b}$, and that is rational since both the numerator and denominator are integers and the denominator is nonzero.

3. How do students realize that $\sqrt{2}+\frac{1}{2}$ is an irrational number?

   Students realize that $\sqrt{2}+\frac{1}{2}$ is irrational when assuming it was rational leads them to a contradiction. Students first began by thinking that $\sqrt{2}+\frac{1}{2}$ is, in fact, rational and from that they wrote $\sqrt{2} = \text{rational number} - \frac{1}{2}$. However, this is a contradiction since the left-hand side is irrational and the right-hand side is rational. Since their steps were correct, the only possible error is the assumption that the original quantity is rational.

4. How do students come up with the conclusion that the sum of any rational and irrational number is always irrational?

   Students in this dialogue pick several examples of a rational number being added to an irrational number. Each time they assume the sum is rational; however, upon rearranging the terms of their equation, they get a contradiction (that an irrational number is equal to a rational number). Since the assumption that the sum of a rational and irrational number is rational leads to a contradiction, the sum must be irrational.

Possible Responses to Related Mathematics Tasks

On odd and even integers

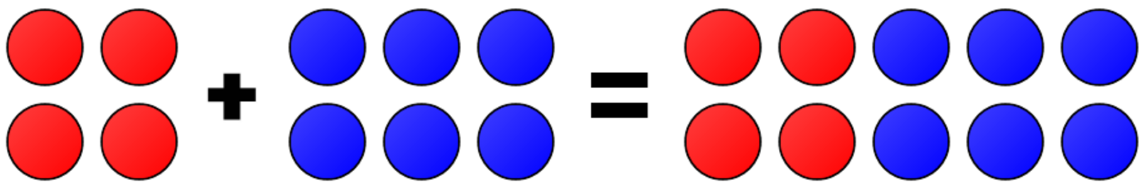
1. For any integer n, what expression can be written to represent all even numbers? What expression can be written to represent all odd numbers? Explain how you came up with your expressions.

   The expression $2n$ can be used to express all even numbers where n is any integer. $2n$ is even since any number doubled is even by definition (it is divisible by 2). The expression $2n+1$ can be used to express all odd numbers since if you pick an even number and count up one integer, you will always end up with an odd number since even and odd integers alternate.

2. Explain why the sum of an even and an even is always even.

   If $2n$ is the first even and $2m$ is the second even, then $2n+2m=2(n+m)$. Since any integer times 2 must be even, that means the sum $(n+m)$ times 2 must also be even.

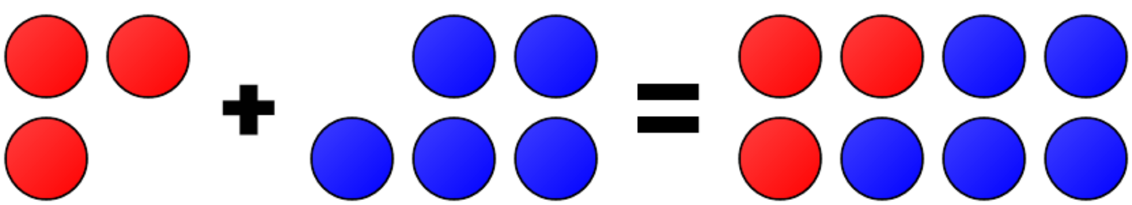
   An alternative explanation can be given using a visual. If numbers are represented by coins, all even numbers could be visualized as a group of coins with two rows. If you take two even numbers represented by coins and put them together (addition), you get a new grouping with 2 rows, which is an even number. An example can be seen below:



3. Explain why the sum of an odd and an odd is always even.

   If $2n+1$ is the first odd and $2m+1$ is the second odd, then $2n+1+2m+1=2(n+m)+2$. We know any integer times 2 must be even, so the sum $(n+m)$ times 2 is even. At the end of the expression, 2 is being added to $2(n+m)$, which keeps the number even because every other integer is even on the number line.

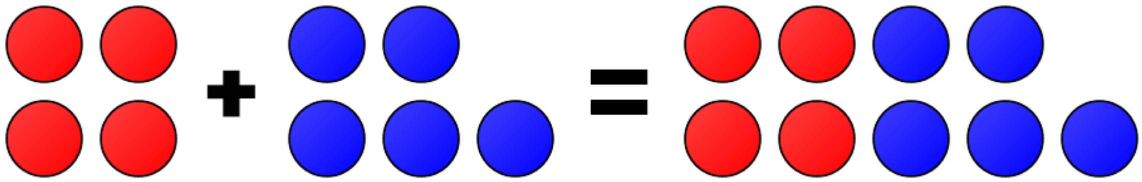
   An alternate explanation can be given using a visual. If numbers are represented by coins, all odd numbers could be visualized as a group of coins with two rows with an extra coin in one of the rows. If you take two odd numbers represented by coins and put them together (addition), you get a new grouping with two complete rows, which is an even number. An example can be seen below:



4. Explain why the sum of an even and an odd is always odd.

   If $2n$ is the even number and $2m+1$ is the odd number, then $2n+2m+1=2(n+m)+1$. We know 2 times an integer is even so $2(n+m)$ is even. However, when you add by 1 you get an odd since evens and odds alternate on the number line.

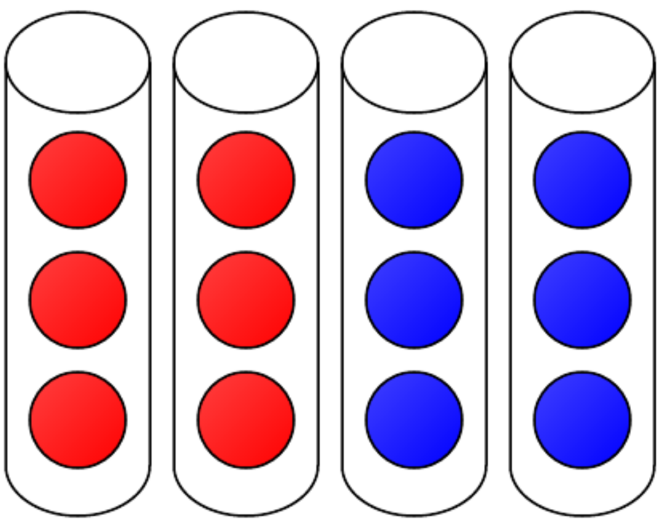
   An alternative explanation can be given using a visual. If numbers are represented by coins, all even numbers could be visualized as a group of coins with two rows, and odd numbers could be visualized as a group of coins with two rows with an extra coin in one of the rows. If you take an even and an odd number represented by coins and put them together (addition), you get a new grouping with two rows plus an extra coin, which is an odd number. An example can be seen below:



5. Explain why the product of an odd and even is always even.

   If $2n+1$ is the odd and $2m$ is the even, then $(2n+1)(2m) = 2 \cdot 2 \cdot (n \cdot m) + 2m$. Since the first term $(n \cdot m)$ is being multiplied by 2, that ensures the first term is even. We already know that the second term is even. Since we are adding two even terms, the sum must be even as shown in question 2.

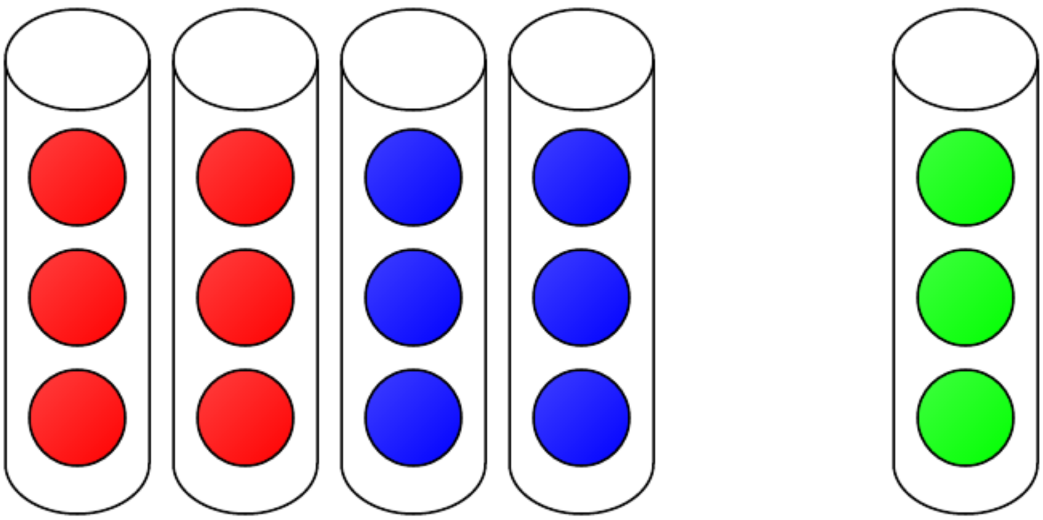
   An alternative explanation can be given using a visual. A product can be viewed as a series of containers with an equal number of coins in each container. In this case, there is an even number of containers with an odd number of coins in each container. However, since there is an even number of containers, you can split the containers into two equal groups. Since each container has the same number of coins, that means the coins can be split into two equal groups so there are an even number of coins. An example can be seen below:



6. Explain why the product of odd and odd is always odd.

If $2n+1$ is the first odd and $2m+1$ is the second odd, then $(2n+1)(2m+1)=2\cdot 2\cdot(n\cdot m)+2n+2m+1$. We know the first term, $2\cdot 2\cdot(n\cdot m)$, is even since we are multiplying by 2, and we know both $2n$ and $2m$ are also even. However, 1 is odd. This means when we add $2\cdot 2\cdot(n\cdot m)$ and $2n$ we get an even. And when we add $(2\cdot 2\cdot(n\cdot m)+2n)$ and $2m$ we get an even again. However, at the end when we add the 1, we end up adding an even and an odd, which gives us an odd.

An alternative explanation can be given using a visual. A product can be viewed as a series of containers with an equal number of coins in each container. In this, case there is an odd number of containers with an odd number of coins in each container. You can split all but one container into two equal groups of containers (and, therefore, all the coins can be split into two groups as well), which means you have an even number of coins in those containers. However, the final container has an odd number of coins in it, and since the sum of an even and an odd number is odd, the total number of coins must be odd. An example can be seen below:



On rational and irrational numbers

7. Find a pair of irrational numbers whose sum is **rational**. Find a pair of irrational numbers whose sum is **irrational**.

   $\sqrt{2}+(-\sqrt{2})=0$ is an example of two irrational numbers whose sum is a rational number.
   $\sqrt{2}+\sqrt{2}=2\sqrt{2}$ is an example of two irrationals whose sum is also irrational.

8. What can you say about the product of two irrational numbers? Support your answer.

   The product of two irrational numbers can be either rational or irrational. For example, $\sqrt{2}\cdot\sqrt{2}=2$ is a case in which two irrational numbers are multiplied together to give you a rational number. However $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$ is a case in which two irrational numbers are multiplied together to give you an irrational number.

9. Is $\sqrt{2} \cdot \frac{1}{2}$ a rational number? Explain why or why not.

No. $\sqrt{2} \cdot \frac{1}{2}$ is an irrational number. This can be seen using reasoning similar to that used in the dialogue. If you rewrite the question as $\sqrt{2} \cdot \frac{1}{2} = \text{rational number}$, this can be rearranged to $\sqrt{2} = \text{rational number} \cdot 2$. Since the right-hand side is a rational number while the left-hand side is irrational, that gives you a contradiction. This means the assumption that $\sqrt{2} \cdot \frac{1}{2}$ is rational must be false and, therefore, $\sqrt{2} \cdot \frac{1}{2}$ is an irrational number.

10. What is the product of a rational and irrational number? Prove your conjecture.

The product of a rational and irrational number is irrational. Consider $z = x \cdot y$ where x is irrational and y is rational and not equal to 0. You can rewrite the equation as $\frac{z}{y} = x$. By substituting several rational numbers for z and y, you can see that x would always come out rational and lead to a contradiction. In general, the quotient of two rational numbers (given the denominator is not 0) is rational. Because a rational z contradicts our definition that x is irrational, this must mean z is irrational. Therefore, the product of a rational and irrational number is irrational.