Absolute Value of Reasoning

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Absolute Value of Reasoning* Illustration: This Illustration's student dialogue shows the conversation among three students who are asked to solve an inequality involving absolute value. As students work toward a solution they use a number line and reason about absolute value as the distance from zero.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 2: Reason abstractly and quantitatively.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 6–7

Target Content Domain: Expressions and Equations, The Number System

Highlighted Standard(s) for Mathematical Content

6.NS.7c	Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of –30 dollars, write* $\lvert -30 \rvert = 30$ *to describe the size of the debt in dollars.*
6.EE.5	Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
6.EE.6	Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

Math Topic Keywords: absolute value, inequality, solve

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Solve the following:

$$|3x - 7| > 14$$

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have been working with absolute value of rational numbers and understand that absolute value is a quantity's distance from zero. They have also solved problems such as $|x| = 4$ *and* $|4x| = 32$*. From previous years, they know the meaning of the inequality symbols. The teacher is challenging them to combine all of this information to solve this task.*

(1) Sam: Okay, so how do we start this? I've never seen anything this hard.

(2) Anita: It's not that hard. Let's think about what the question is asking… absolute value of $3x - 7$ is greater than 14.

(3) Sam: I know absolute value means a number's distance from zero, but what does that mean here?

(4) Anita: Remember when we did things like this? *[writes* $|2x|=6$*]* We had to think about what numbers make that true. So we said that whatever is inside the bars has to be 6 away from 0?

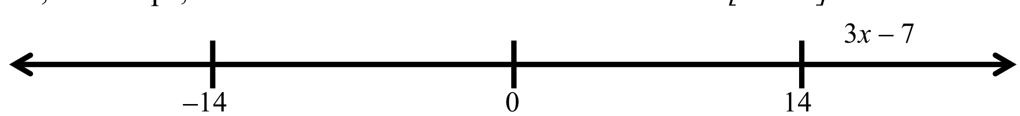
(5) Sam: Yeah, I remember and we got that it was positive and negative.

(6) Anita: Yeah, we got positive and negative *usually.* So we just have to think about that.

(7) Sam: Okay, but this time we have greater than, not equal to.

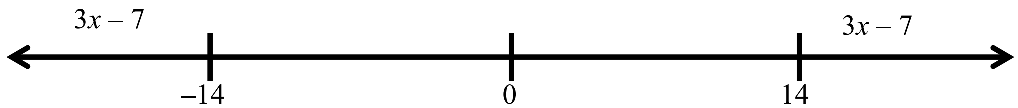
(8) Dana: So, what this really says is that this *[pointing to* $3x - 7$*]* is more than 14 from zero.

(9) Sam: Oh, that helps, Dana. So here's what I think it looks like *[draws]*…



And what's inside the lines, $3x - 7$, has to be over here somewhere. *[points to the space to the right of 14]*

(10) Anita: Yeah, that’s right, but also down here. *[points to the space left of -14]*



(11) Sam: Ok, yeah. But again, how does that help us?

[Students pause to think.]

(12) Anita: What if we figure out what makes $3x - 7$ *equal to* 14.

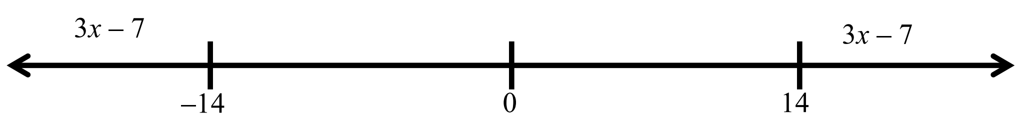
(13) Dana: If something minus 7 is 14, then that something has to be 21, so $3x$ is 21 which means x is 7.

(14) Anita: Okay, so if 7 makes it equal 14, then I think if x is bigger than 7 that must make the whole thing bigger than 14. Let’s try 8. *[calculates]* Yup, it works.

(15) Sam: So 8 and higher work; does that look like $x > 8$?

(16) Anita: No, we said anything bigger than 7, so we have to include $7\frac{1}{2}$ and numbers like that. That’s $x > 7$.

(17) Dana: That takes care of the ones up here *[points to right of 14 on Sam’s original number line]*, but what about these down here? *[points to left of –14]*



(18) Anita: That’s what I meant when I said that $3x - 7$ could be down here. *[points to the number line left of –14]*

(19) Sam: So wouldn’t it be just $x < -7$?

[Students think for a minute.]

(20) Dana: If x is –7, then $3x - 7$ is –28, which fits, but that’s way less than –14, so there must be other solutions, too.

(21) Anita: Something happened when 7 made $3x - 7 = 14$, so let’s make $3x - 7 = -14$ and see what that does. *[thinking out loud]* If something minus 7 is –14, that’s $-7 - 7$, so $3x$ has to be –7. If 3 times something has to be –7, then that’s $-7 \div 3$ which is…

(22) Sam: –2.33333333333333333333…

(23) Anita: OR... $-\frac{7}{3}$.

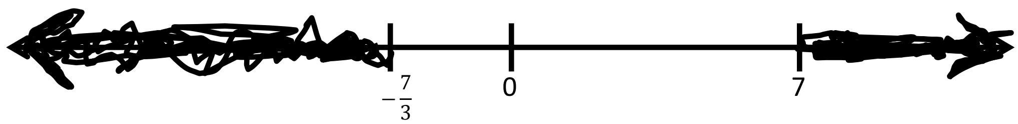
(24) Sam: Yeah, that too. But now what?

(25) Dana: So, now we know that since $-\frac{7}{3}$ makes it equal –14, anything less than that makes it less than –14. Isn't that what we want?

(26) Anita: I think so, and that looks like $x < -\frac{7}{3}$.

(27) Sam: Can't we check? –3 is less than $-\frac{7}{3}$. And if we use –3 for x, then $3x - 7$ is *[calculates]* –16. Booyah!

(28) Anita: So, on the number line it looks like this, right? *[draws]*



(29) Sam: Hey, wait! Where did the 14s go?

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

2. In line 29, Sam asks "Where did the 14s go?" What does this question mean and how would you answer it?

3. One of the aspects of this problem that makes it a bit complex is the coefficient of 3 in $3x$. One of the ways to deal with this is to divide everything by 3. Will this work with an absolute value inequality? Why?

4. In the Student Dialogue, students use the meaning of absolute value as the distance from zero. However, the expression $|3x-7|$ could also be interpreted as the distance between $3x$ and 7. How would this change the reasoning students might use?

5. If you were lingering at the table listening to this group of students in this Student Dialogue, is there any place you would interject? Where and what would you say or do?

Absolute Value of Reasoning

Mathematical Overview

Suggested Use The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
Reason abstractly and quantitatively.	There are two places in the Student Dialogue where the students are reasoning about this particular problem. The first comes in lines 4 through 9 where they are reasoning about the absolute value of $3x - 7$ and “attending to the meaning of the quantity” and the inequality in relation to their understanding of absolute value. This reasoning also helps them make sense of the problem (MP 1). They also use this reasoning when they transition from equation to inequality in lines 13 and 25. The second place they use MP 2 is in line 21 when Anita calculates (out loud) the solution for $3x - 7 = 14$ “as though [the symbols] have a life of their own.”
Construct viable arguments and critique the reasoning of others.	Much of this Student Dialogue is “mak[ing] conjectures and build[ing] a logical progression of statements to explore the truth of their conjectures.” When Anita poses (line 14) that “7 makes it equal 14, then I think if *x* is bigger than 7, that must make the whole thing bigger than 14,” she conjectures and together they explore that idea. And in lines 25–27, they make more conjectures and prove themselves correct. In line 16, Anita critiques (and corrects) Sam’s statement that bigger than 7 means 8 and larger, and reminds him to consider numbers other than the integers.
Use appropriate tools strategically.	A *very* helpful tool for the students in this Student Dialogue is the number line. Used in two different situations through the problem solving, the number line helps the students illustrate their thinking as well as understand each other. The first helps them make sense of the original inequality, while the second becomes a representation of the solutions to the inequality.
Look for and make use of structure.	There are two strong uses of structure in this Student Dialogue that might be overlooked as using structure. The first is in line 8 when Dana points to $3x - 7$ and calls it “this.” She is treating the “algebraic expression as a single object” and declares where it must be on the number line. The second use of structure also deals with $3x - 7$ in line 21, when Anita sees the expression as “something minus 7 is –14” and determines that *something* must be –7 and moves on to find the value of *x*. The same line of thinking is used by Dana in line 13. Sam is still working on finding and using structure.

Absolute Value of Reasoning

Commentary on the Mathematics

There are a few assumptions and definitions in this Illustration that are important to bring to the surface. One is the meaning of "solution" or the process of solving an equation or inequality. As stated in the content standard (6.EE.5), a solution for an equation or inequality is the set of values that makes the statement true. In some instances, this solution set contains only one value ($3x + 7 = 4$). In other instances, there might be 2 or 3, (e.g., $x^2 + 7x + 12 = 0$, $|x| = 7$) or some other finite number. There are others for which the solution set contains no values ($3x + 7 = 3x + 9$) and many for which the solution set contains an infinite number of values (or pairs or groups of values). Whatever the particular case, it is important that students are in the habit of reasoning about what a solution set is. The graph of a solution set is also an important idea for students to understand well. One definition of the word "graph" is a picture. Using this, the graph of a solution set is a visual representation of that set. Being able to make the connection between a solution set and its graph is one way for students to reason through a problem and its particular constraints. In this illustration, the graph—the number line—was used by the students to help them understand the problem and reason through to a solution set. A picture can be worth a thousand words…

A second assumption at hand is that an absolute value function is continuous across the domain. This is something that is inherently accepted by the students and, at this level, that's sufficient. This idea plays out, without the students realizing it, in their declarations that everything in a region is a solution (or is not a solution) to the inequality. This (accurately) presumed continuity is also important to the idea that this inequality creates three regions out of the domain.

Beyond the assumptions and definitions, there are two main mathematical ideas at play in this Illustration:

1. Chunking—seeing an expression as a single object, e.g., seeing $|3x - 7| > 14$ as $|\bigstar|>14$—as Dana does in line 8 when using the word "this"
2. Connecting algebra and geometry—interpreting absolute value as distance

The use of chunking in this particular Illustration is the way students approach the problem, and the detailed description of it is included in the evidence of MP 7. This practice of chunking is very useful as students advance in their mathematical education and experience. Particularly helpful in algebraic reasoning and manipulation, seeking an expression or portion of an equation that can be "chunked" can transform the expression into a familiar form that we know how to address. The students demonstrate this in the Student Dialogue in lines 8 and 9, which allows them to understand that the absolute value of "this" being greater than 14 means that "this" is more than 14 units from zero in either direction on the number line.

The interpretation of absolute value as distance deserves a little more attention. We often calculate the distance between two points, a and b, on the number line by calculating $|a - b|$. This is a more generalizable definition of absolute value because we can use it for any values of a and b, and from this $|a|$ can be viewed as $|a - 0|$, which is the distance from a to 0 (the more widely understood definition of absolute value). This definition of distance from 0 is the one that students turn to in this example, and they reason through the problem to find the solution set. If they were to use the alternative perspective, they would have come to the same solution (see Teacher Reflection Question 4 for further information). There are other advantages to using the broader definition as well:

- By using this perspective, we can eliminate the confusion that many students experience around the algebraic definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

  It is important to note that Sam, Dana, and Anita very deftly avoid this whole confusion through their approach of combining chunking and the perspective of absolute value as a distance. The other often-confusing rule they avoid is the "rule" that when solving an inequality, if you multiply or divide by a negative number, you have to switch the sign. This doesn't even come into their conversation. By reasoning about the distance that $3x - 7$ must be from zero, they inherently switch the sign without the oft-memorized rule.

- This definition extends to other contexts. For example, on the complex plane, if $|z|$ is defined as the distance from z to 0, then the equation of the unit circle becomes $|z| = 1$.

The interpretation of absolute value as a distance (either from zero or between two quantities) has applications all over algebra and analysis. For example, students often use formal algebra to convert a repeating decimal to a fraction. In this illustration, Sam and Anita agree that

$$-2.33333333333... = -\frac{7}{3}$$

One way to show this is to argue as follows:

$$\text{Let } s = -2.3333333\ldots$$

Then

$$10s = -23.333333$$

Subtracting the first from the second

$$9s = -21$$

$$s = -\frac{21}{9} = -\frac{7}{3}$$

But there is something subtle about the reasoning here because it can lead to strange results. Suppose

$$r = 1 + 2 + 2^2 + 2^3 + \cdots$$

Intuitively, since r is the sum of all of the powers of 2, it is infinitely large. However,

$$r = 1 + 2 + 2^2 + 2^3 + \cdots$$

$$2r = 2 + 2^2 + 2^3 + 2^4 + \cdots$$

and then, subtracting the first from the second,

$$r = -1$$

Absolute Value of Reasoning

How can it be that r is infinity *and* –1? The paradox demonstrates the need to consider distance—absolute value—when thinking about infinite sums and limits. In the Illustration titled *Factoring a Degree Six Polynomial*, we consider a version of the identity

$$(1-x)(1+x+x^2+x^3+\cdots+x^{n-1})=1-x^n$$

This is a formal identity, so it holds true for any real value of x. One way to look at this is to multiply all the terms by $(1-x)$ and watch the terms fall away. A similar formal calculation suggests that

$$(1-x)(1+x+x^2+x^3+\cdots x^{n-1}+x^n+\cdots)=1$$

Because, in this case, *all* of the terms fall away except for 1. This leads to the identity

$$(1+x+x^2+x^3+\cdots x^{n-1}+x^n+\cdots)=\frac{1}{1-x}$$

In this case, if we let $x = 2$, we get our strange result that the sum of the powers of 2 is –1. The problem is that this identity, when interpreted as a statement about numbers rather than about formal expressions, is really shorthand for a limit. For a number x, the claim is that

$$\lim_{n\to\infty}(1+x+x^2+x^3+\cdots x^{n-1}+x^n+\cdots)=\frac{1}{1-x} \quad (*)$$

But this is only true for *some* values of x. For any finite value of n,

$$(1+x+x^2+x^3+\cdots x^{n-1})=\frac{1-x^n}{1-x}$$

The right-hand side of this has a finite limit if x^n has a finite limit as n goes to ∞. This only happens when x is within 1 of 0 – i.e. when $|x|<1$.

So, Sam and Anita's claim that $-2.333\ldots = -\frac{7}{3}$ can be made precise by writing the infinite decimal as an infinite sum. First, remember that

$$.333\ldots=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots=\frac{3}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)=\frac{3}{10}\left(1+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)$$

using this notation

$$-2.333\ldots=-\left(2+\frac{3}{10}\left(1+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)\right)$$

Because $\left|\frac{1}{10}\right| < 1$, we can invoke the identity * which tells us that

$$1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}$$

and substituting this in, we get that

$$-2.333... = -\left(2+\frac{3}{10}\left(\frac{10}{9}\right)\right) = -\frac{7}{3}$$

In contrast to this, the sum $1 + 2 + 2^2 + 2^3 + \ldots$ doesn't converge at all because $|2| > 1$.

This isn't the end of the story, though. There are other ways to define distance—and absolute value—that have been shown to have applications throughout mathematics. In one such system, $|2|$ is indeed less than 1, so the infinite sum $1 + 2 + 2^2 + 2^3 + \ldots$ does converge to –1. For details on this, see "Making a Divergent Series Converge" in *Mathematics Teacher* (December 1984).

<u>Evidence of the Content Standards</u>
In the CCSS, 6th grade is where students are expected to solidify their understanding of absolute value of rational numbers (6.NS.7c), which the students in the Student Dialogue discuss in lines 3 and 4. This task asks the students to extend that understanding to a variable expression, which is also a prominent part of 6th-grade content standards (6.EE.6). The understanding of solving an equation or inequality as finding a set of values that make the statement true (6.EE.5) comes into play in lines 14–16, 19, and 28. (See previous section for more on this.)

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. On the second number line in the dialogue, the labeled values are $-\frac{7}{3}$ and 7. Where did those values come from? Should these values be included in the solution or not?

2. In line 29, Sam says "Where did the 14s go?" What does this question mean and how would you answer him?

3. In line 19, Sam says "wouldn't it just be $x < -7$? Why does Sam say that?

Related Mathematics Tasks

1. If you know that the solutions to an absolute value equation are 6 or –6, what might have been the equation?

2. If you know the solutions to an absolute value equation are 7 or –5, what might have been the equation?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. In line 29, Sam asks "Where did the 14s go?" What does this question mean and how would you answer it?

   Sam is asking why 14 and –14 are no longer significant points on the second number line like they were on the first. There are a few ways to think about this. In the first inequality, the students interpret the absolute value as a quantity's distance from zero. This quantity is $3x - 7$, and its distance from zero is greater than 14 units. So the first graph is a visual representation of that understanding. Here 14 and –14 are important. The second graph represents the values of x that will make the original inequality true. Even though the two are related, the graphs represent very different things. This is one way to think about it. For more discussion, see the Mathematical Overview.

3. One of the aspects of this problem that makes it a bit complex is the coefficient of 3 in $3x$. One of the ways to deal with this is to divide everything by 3. Will this work with an absolute value inequality? Why?

   Short answer is yes. If we use the idea that absolute value represents a quantity's distance from zero, then $\frac{1}{3}$ of that quantity would be $\frac{1}{3}$ of the distance from zero (i.e., if x is 9 units from 0, then $\frac{1}{3}x$ is 3 units from zero). As for the inequality, if something is *more* than 3 units from 0, then it is either further to the right or further to the left. So, as long as the computation is done to both sides of the inequality, then division by a non-zero number is mathematically correct. In the case of this task, division by 3 would make the inequality read $\left|x - \frac{7}{3}\right| > \frac{14}{3}$.

4. In the Student Dialogue, students use the meaning of absolute value as the distance from zero. However, the expression $|3x-7|$ could also be interpreted as the distance between $3x$ and 7. How would this change the reasoning students might use?

   If $|3x-7|>14$ is taken to mean that the distance between $3x$ and 7 is greater than 14, that must means $3x$ has to be more than 14 away from 7. This gives us two possibilities: either $3x$ is larger than 21 (more than 14 above 7) or $3x$ is smaller than –7 (less than 14 below 7). This leads to the solution $x>7$ or $x<-\frac{7}{3}$, just like those found in the Student Dialogue.

5. If you were lingering at the table listening to this group of students in this Student Dialogue, is there any place you would interject? Where and what would you say or do?

   Discuss with colleagues to share ideas.

Possible Responses to Student Discussion Questions

1. On the second number line in the dialogue, the labeled values are $-\frac{7}{3}$ and 7. Where did those values come from? Should these values be included in the solution or not?

   These are the values that solve the equations $3x-7=14$ and $3x-7=-14$ and are the points beyond which $3x-7$ is more than 14 units away from zero. Sometimes we refer to these as "cutoff points" or "boundary points." Between these points, $3x-7$ is less than 14 units away from zero. They should not be included in the solution because the inequality doesn't include "equal to."

2. In line 29, Sam says "Where did the 14s go?" What does this question mean and how would you answer him?

   Sam is asking why 14 and –14 are no longer significant points on the second number line like they were on the first. In the first inequality, the students interpret the absolute value as a quantity's distance from zero. This quantity is $3x-7$, and its distance from zero is greater than 14 units. So the first graph is a visual representation of that. Here 14 and –14 are important. The second graph represents the values of x that will make the original inequality true. 14 and –14 are still on the number line and included in the solutions, but they aren't critical values anymore.

3. In line 19, Sam says "wouldn't it just be $x<-7$? Why does Sam say that?

   Often, when we solve absolute value equations, the solutions are a positive and negative of the same number (+7 or –7; +12 or –12). However, this is only true if the equation is $|x|=7$ or something like that. Since Anita just said that a solution to the original inequality is $x>7$, Sam mistakenly applies this kind of thinking to say that $x<-7$.

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Possible Responses to Related Mathematics Tasks

1. If you know that the solutions to an absolute value equation are 6 or –6, what might have been the equation?

   $|x| = 6$ or something equivalent such as $|3x| = 18$ or $|x| + 8 = 14$.

2. If you know the solutions to an absolute value equation are 7 or –5, what might have been the equation?

   $|x - 1| = 6$. This can be viewed as a transformation of the equation in question 1, but there is another way to consider it as well. If we use the understanding that absolute value represents the distance between the two terms within the bars, we can work this backwards from the solution. In other words, if $|a - b| = c$, then we know that a and b are c units apart. This, in turn, means that a could be c units to the right or left of b. In the specific case posed, know that x is either 7 or –5 and that it is equidistant from some other number (let's call it b). Because 7 and –5 are equidistant from 1, then we know b is 1. We also know the distance is 6 because we can calculate it. So, we can say that the distance from x to 1 is 6, which translates to $|x - 1| = 6$.

   Try this one now… if you know the solutions to an absolute value equation are 29 and 5, what could be the equation?