

Consecutive Sums

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Consecutive Sums* Illustration: This Illustration’s student dialogue shows the conversation among three students who are asked to think about how many different ways a number can be written using consecutive sums. As they explore different consecutive sums students are able to rewrite sums as products and find that any multiple of an odd number can be written as a consecutive sum.

Highlighted Standard(s) for Mathematical Practice (MP)

MP 1: Make sense of problems and persevere in solving them.

MP 7: Look for and make use of structure.

MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 6–7

Target Content Domain: The Number System, Expressions & Equations

Highlighted Standard(s) for Mathematical Content

- 6.NS.C.6a Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.
- 6.EE.A.3 Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2+x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*
- 7.NS.A.1b Understand $p + q$ as the number located a distance $|q|$ from p , in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
- 7.EE.A.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a + 0.05a = 1.05a$ means that “increase by 5%” is the same as “multiply by 1.05.”*

Math Topic Keywords: consecutive numbers, sums, factors

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Mathematics Task

Suggested Use

This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

The number 13 can be expressed as a sum of consecutive positive integers $6 + 7$. Fourteen can be expressed as $2 + 3 + 4 + 5$, also a sum of consecutive positive integers. Some numbers can be expressed as a sum of consecutive positive integers in more than one way. For example, 25 is $12 + 13$ and is also $3 + 4 + 5 + 6 + 7$. Look for evidence that would be useful in answering the questions: Is there a general rule for how many different ways a number can be written as a sum of consecutive positive integers? If there is a general rule, what is it?

Task Source: Problem adapted from Benson, S., Addington, S., Arshavsky, N., Cuoco, A., Goldenberg, E. P., & Karnowski, E. (2005). *Ways to think about mathematics*. Thousand Oaks: Corwin Press.

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Student Dialogue

Suggested Use

The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

This conversation occurs during the second day of working on this problem. Students have already made some discoveries about the problem and shared some insights during the previous day's class. In particular, one group noticed that the sum of any three consecutive integers is equivalent to three times the middle number. Students begin with this idea on the second day.

- (1) Dana: We know a way to find the sum of any three consecutive numbers. We know $7 + 8 + 9$ without even adding. We can just multiply: 8 times 3 is 24.
- (2) Sam: I think it works because the 7 and 9 sort of balance each other out—one of them is 1 less than 8 and the other is 1 more than 8.
- (3) Anita: So, let's write it like this: *[writes]* $(8 - 1) + 8 + (8 + 1)$.
- (4) Sam: Yeah! That's what I said, but I like how you wrote it, Anita. You can see that the negative and positive 1s cancel out so you're left with $8 + 8 + 8$.
- (5) Dana: I like it! What if we take it further?
 $(8 - 2) + (8 - 1) + 8 + (8 + 1) + (8 + 2)$
The negatives and positives cancel out again, so $6 + 7 + 8 + 9 + 10$ must be 8 times 5, which is 40.
- (6) Sam: And one step further would be
 $(8 - 3) + (8 - 2) + (8 - 1) + 8 + (8 + 1) + (8 + 2) + (8 + 3)$
That's $5 + 6 + 7 + 8 + 9 + 10 + 11$, or 8 times 7, which is 56. So, we've made $8 \cdot 3$, $8 \cdot 5$, and $8 \cdot 7$.
- (7) Anita: But there's nothing special about 8. Can we try this with another number like 11?
- (8) Dana: Sure. $11 \cdot 3 = 33$ and we can make 33 with $(11 - 1) + 11 + (11 + 1)$, which is $10 + 11 + 12$. Yup, that checks.
- (9) Anita: And $11 \cdot 5 = 55$, and that's... well, I don't want to write it out. But we need 5 numbers, 2 of them lower than 11 and 2 higher, so it starts with $11 - 2$ and ends with $11 + 2$. So we're adding the numbers between 9 and 13: that's $9 + 10 + 11 + 12 + 13$. That's... 55.
- (10) Sam: And $11 \cdot 7 = 77$ and that sum starts with $11 - 3$ and ends with $11 + 3$. So that's $8 + 9 + 10 + 11 + 12 + 13 + 14$. I'm going to assume that's 77. I mean it *has* to be because it's the same as adding seven 11s together.

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(11) Dana: Great, so now we can say that the sum of three consecutive numbers is 3 times the middle number. The sum of five consecutive numbers is 5 times the middle number. And the sum of 7 consecutive numbers is 7 times the middle number.

(12) Anita: That's true, Dana. But instead of saying it that way, what if we look at it this way: we just showed a way to make *any* multiple of 3, *any* multiple of 5, and *any* multiple of 7. I think we can even say that we can make any multiple of any odd number greater than 1.

(13) Sam: Whoa! I need to think about that for a minute.

[Students pause and think on their own.]

So if we can make *any* multiple of *any* odd number, can we also make any multiple of any even number?

(14) Anita: Wait... Maybe we can think about that in a minute, Sam. I think I'm onto something else: What if a number is the multiple of 2 odd numbers? Take 15: it's a multiple of 3 and a multiple of 5.

Since we can make any multiple of 3 with 3 numbers and any multiple of 5 with 5 numbers, that means we have two different ways of making 15:

$$15 = 5 + 5 + 5 = (5 - 1) + 5 + (5 + 1) = 4 + 5 + 6 \text{ and}$$
$$15 = 3 + 3 + 3 + 3 + 3 = (3 - 2) + (3 - 1) + 3 + (3 + 1) + (3 + 2) = 1 + 2 + 3 + 4 + 5$$

(15) Dana: Yeah, that makes sense. Let me try it. I'll use 33 because we already found one of the sums: $10 + 11 + 12$. So now if we think of this as a multiple of 11, then 3 is the middle number. Since we need 11 numbers, the biggest number is $3 + 5$ and the smallest number is $3 - 5$... Wait.

(16) Sam: We can't use negative numbers.

(17) Anita: Sam's right. We're only supposed to use positive number integers in this problem. But there's nothing wrong with the math. It should still work out... Try it, Dana.

(18) Dana: Ok. So we have: $(3 - 5) + (3 - 4) + (3 - 3) + (3 - 2) + (3 - 1) + 3 + (3 + 1) + (3 + 2) + (3 + 3) + (3 + 4) + (3 + 5)$. That's 11 numbers. It has to be $3 \cdot 11$ because we know all the positives and negatives cancel out.

(19) Sam: That means 33 is $-2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. So I guess we just extended the problems to include negative numbers. I'll check it. -2 plus -1 is -3 ...

(20) Anita: Wait, there's a shortcut. The positives and negatives cancel again! The first five numbers add up to 0, so 33 is equal to the sum of the last six numbers: $3 + 4 + 5 + 6 + 7 + 8$. Right?

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(21) Sam: Wow. That worked! How else can we write 33? What about $1 \cdot 33$? Um... Never mind. That's too many numbers.

(22) Dana: Wait. Let's try it. I bet we can find some shortcuts.

(23) Sam: Ok. So $1 \cdot 33$ is $1 + 1 + 1 + \dots$. You know, 33 times.

(24) Anita: Right. And out of those, we want the middle number to be 1.

(25) Sam: So half of the rest of the numbers are below... We started with 33 numbers, but we know the middle number, so that leaves 32 numbers. Half of that is 16. So there are 16 numbers below and 16 numbers above! So we have

$$\underbrace{1+1+\dots+1}_{16\text{times}}+1+\underbrace{1+1+\dots+1}_{16\text{times}}$$

(26) Dana: So the lowest number is $1 - 16$ and the highest number is $1 + 16$. So we're adding all the numbers from -15 to 17 .

(27) Sam: But we *also* know the sum of the integers from -15 to 15 is 0 because all the positives and negatives cancel out. So that just leaves $16 + 17$... Yup, that's 33. That wasn't so bad.

(28) Anita: So we found three ways to write 33: as a multiple of 3, a multiple of 11, and a multiple of 33. That was $10 + 11 + 12$, then $3 + 4 + 5 + 6 + 7 + 8$, and $16 + 17$. Are there any more?

[Students work for 10 minutes.]

(29) Sam: I don't think so. There aren't too many more options to work with. If there *were* another way to write 33 as a sum, it would have to be a sum of 4 or 5 positive numbers. I already looked at adding 1 through 7 and that didn't give us 33 and then adding 1 through 8 is too big.

(30) Dana: It can't be a sum of 5 numbers, because it's not a multiple of 5.

(31) Sam: Right. But we said the multiple rule only applies to odd numbers, so we can check the sums of 4 numbers, like $5 + 6 + 7 + 8$.

(32) Anita: Well, we already know that doesn't work.

(33) Sam: Oh yeah, that's already part of one of our sums. I meant $6 + 7 + 8 + 9$. Actually, we can check those using our sums, too. We know that $3 + 4 + 5 + 6 + 7 + 8$ is 33, so $6 + 7 + 8 + 9$ can't be, because $3 + 4 + 5$ is bigger than 9. Anyway, none of the sums of 4 numbers I tried worked, so there can't be any more ways to write 33.

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(34) Anita: Great! So we've figured out 33. And we got there by looking at multiples of odd numbers. Do you think there's a connection between factors and sums in general?

(35) Dana: Let's figure it out.

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Teacher Reflection Questions

Suggested Use

These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of the students in the dialogue engaging in the Standards for Mathematical Practice?
2. In this Illustration, we drop in on students as they work on this problem for the second day. What might the first day's conversation have looked like? How might you steer the first day's discussion so that students might have conversations like this one on the second day?
3. In what direction might the students take their further exploration of this problem?
4. In lines 29–33, the students look for another way to write 33 as a sum of consecutive numbers besides the ones they have already found. How do they know they only have to look for sums of 4 or 5 numbers?
5. It is impossible for 33 to be written as a sum of 4 consecutive numbers. Without trying to calculate them, it is also possible to tell immediately that 195, 2911, 77, and 243 can't be written as sums of 4 consecutive numbers, either. Why not?
6. In line 13, Sam asks “So if we can make *any* multiple of *any* odd number, can we also make any multiple of any even number?”
7. What evidence is there that the students will or will not get to the generalization that the task requests—finding a general rule for how many different ways a number can be written as a consecutive sum? What questions might you ask to guide them?
8. What are some ways to modify the original problem to produce variations that may also provide some interesting opportunities to investigate numbers?

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Mathematical Overview

Suggested Use

The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 <p>Make sense of problems and persevere in solving them.</p>	<p>In this Illustration, we do not see the initial attempts at entry into the problem, since we enter the conversation on the second day. Instead, we see evidence of perseverance in ways that the students “analyze... relationships and goals,” “make conjectures about the form and meaning of the solution,” and “try simpler cases... in order to gain insight into its solution.” Students begin the Student Dialogue with the observation that given any three consecutive numbers, the sum will be triple the middle number (line 1). By itself, this observation gives only a shortcut for addition, not any special insight into the problem, until the students analyze the relationship more deeply. They examine the structure and make a generalization (lines 2–11). Then in line 12, we see that Anita takes this generalization and rewords it in a way that helps the group move towards their goal. Anita’s action of taking a result the group has established and recasting it in a way that advances the group towards a solution demonstrates MP 1.</p> <p>Anita’s observation in line 12 leads to two new conjectures: an underformed conjecture begun by Sam (line 13) about even multiples, and a conjecture by Anita and Dana (lines 14–15) about generating multiple ways of writing a number as a sum. As Dana starts to test her conjecture, the students quickly run into another opportunity to demonstrate perseverance. In lines 15–16, the students see that their “tried and true” method is leading them to use negative numbers, though the problem specifies the sums are of consecutive positive integers. Because the students have examined “analogous problems” and have convinced themselves that their process makes sense, Anita is able to say in line 17, “But there’s nothing wrong with the math. It should still work out...” and the students persevere and eventually find that their process results in a sum of positive integers (line 20).</p>
 <p>Look for and make use of structure.</p>	<p>The solution pathway explored by the students is largely informed by structure. In line 2, Sam supports the idea that the sum of three consecutive numbers should be three times the middle number by saying this makes sense because one number is 1 less than the middle number while the other is 1 greater. Anita expresses this in writing in line 3 by</p>

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	<p>rewriting $7 + 8 + 9$ as $(8 - 1) + 8 + (8 + 1)$. This action, re-expressing a quantity in a way that makes the underlying structure visible, is an important way that students demonstrate MP 7.</p> <p>Identifying the structure of the sum of three consecutive sums in this way allows students to extend that structure to the sum of five, seven, and any odd number of integers (lines 5–12). We see that Sam is convinced of this structure in line 10 when he sets out to verify that 77 is $8 + 9 + \dots + 14$, but realizes he doesn't need to verify it because he understands that adding the integers from $(11 - 3)$ to $(11 + 3)$ <i>has</i> to be the same as adding seven 11's together.</p> <p>The students extend the structure to include the sum of negative integers in lines 18–20. Though the initial problem specifies that the sums must be of positive integers, the students see that, for the underlying structure to be consistent, they need to use negative integers. Happily, they also find that the negative integers seem to cancel out (lines 20, 27) in these cases.</p> <p>We also see evidence of MP 7 when students reason that since they have shown that they can write any multiple of an odd number using consecutive sums (line 12), they must be able to look at a number's factors to find new consecutive sums (lines 14, 15, 21). The students "step back for an overview and shift perspective."</p>
 <p>Look for and express regularity in repeated reasoning.</p>	<p>We see several instances where "students notice if calculations are repeated, and look both for general methods and for shortcuts." For example, after examining and extending the structure to write sums for $8 \cdot 3$, $8 \cdot 5$, and $8 \cdot 7$ (lines 1–6), the students repeat the reasoning to write sums for $11 \cdot 3$, $11 \cdot 5$, and $11 \cdot 7$ (lines 8–10). In their reasoning for multiples of 11, furthermore, the students use shortcuts (line 9: "I don't want to write it out...[but] it starts with $11 - 2$ and ends with $11 + 2$"). Evidence that students are demonstrating MP 8 is not just in the repetition, but also because they use shortcuts and use their work to make general statements (lines 11–12). We see a similar process when, after the students work through writing the $11 \cdot 3$ as the sum of 11 numbers (lines 15–20), they repeat the process to write $33 \cdot 1$ as the sum of 33 numbers (lines 21–27). By the end of the Illustration, we see that students have begun to develop a general method for finding all the possible sums for any given number, but we don't see the full generalization (for example, they are not able to show that they have found all the ways to write 33 as a sum except by some guess-and-check of different sums of 4 numbers). We would expect to see further evidence of MP 8 in whatever process the students use to figure out the general connection between factors and sums.</p>

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Commentary on the Mathematics

This problem is an example of an open-ended problem with a low entry point—allowing access for all students—which then tackles a lot of interesting mathematics. All students can begin to engage with the problem by simply starting to add consecutive numbers and recording their solutions. This problem is also an example of one that benefits from students making a solution pathway. Though students may begin by exploring the different sums possible from consecutive counting numbers, they will find that discernible patterns only emerge from a methodical treatment of the sums. This problem also allows for multiple solution pathways. Students must consider imposing additional constraints to create sub-problems, and can do this in several ways. Finally, this problem is a long problem that requires students to consider several cases before starting to formulate any sort of solution. Students must demonstrate perseverance and make informed decisions about what solution pathway to pursue, and how to know when they have reached a conclusion that they can carry forward.

This rich mathematical problem can be explored at many different levels. In this commentary, we further develop the ideas of the Student Dialogue into a proof for those interested. Also, please note that there are other approaches to this problem and other conclusions that may be reached that are not demonstrated here.

Theorem: The number of ways a positive integer can be written as a consecutive sum is one fewer than the number of odd factors it has. For example, 9 has three odd factors (i.e., 1, 3, 9) and there are 2 ($= 3 - 1$) ways a consecutive sum can be written for 9 (i.e., $2 + 3 + 4$ and $4 + 5$).

Proof: First, note that the students have demonstrated that every odd factor leads to a way of writing a number (which will be referred to as the “target” number) as a consecutive sum.

A sum may contain either an even number or odd number of summands.

If there is an odd number of summands, we know that the sum (the target number) can be written as a product of the number of summands (the odd factor) and its partner factor (the middle term of the string of summands). For any given odd factor, the sum (with its odd number of summands) will be unique. That is, a sum of one set of five consecutive numbers cannot equal the sum of a different set of five consecutive numbers.

If there is an even number of summands, we can show that these must *also* be related to an *odd* factor of the target number:

- First note that every sum of positive consecutive numbers can be written as a sum of consecutive numbers that includes non-positive numbers. Suppose a is positive and is a part of the consecutive sum $a + (a + 1) + \dots + (a + x)$. This sum will equal the sum $-(a - 1) + -(a - 2) + \dots + -1 + 0 + 1 + \dots + (a - 2) + (a - 1) + a + (a + 1) + \dots + (a + x)$.
- The sum $-(a - 1) + -(a - 2) + \dots + -1 + 0 + 1 + \dots + (a - 2) + (a - 1)$ will always contain an odd number of summands.
- So now suppose that the original consecutive sum $a + (a + 1) + \dots + (a + x)$ has an even number of summands. Then we can rewrite it in its longer form that includes non-positive numbers, and it will have an odd number of summands. The middle summand of this longer

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form will always be ≥ 0 (because there will always be more positive summands than negative summands). And any sum that has an odd number of summands can be rewritten as a product of the target number with an odd factor.

So every possible consecutive sum that can be written can be related to an odd factor of the target number.

Now all that remains is to show that *every* odd factor produces a unique sum. This is straightforward. Every odd factor will produce a sum of a different length (with a different number of summands). Note that it is impossible for two different sums to be “shortened” into the same sum. Given a sum of positive summands $a + (a + 1) + \dots + (a + x)$, the longer sum $-(a - 1) + \dots + (a - 1) + a + (a + 1) + \dots + (a + x)$ is unique.

Every number has a factor of 1, which produces a “consecutive sum” of length 1. For the purposes of this problem, those consecutive sums with just a single summand (e.g., powers of 2) will not be counted. So the number of ways a positive integer can be written as a consecutive sum is one less than the number of odd factors it has. (QED)

Evidence of the Content Standards

The mathematical task and Student Dialogue are related to two content standards in grade 6 and two in grade 7. In grade 6, students must “recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line” (6.NS.C.6a) and “apply the properties of operations to generate equivalent expressions” (6.EE.A.3). In grade 7, students must “show that a number and its opposite have a sum of 0 (are additive inverses)” (7.NS.A.1b) and “understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related” (7.EE.A.2). In this Illustration, students rewrite the sum $7 + 8 + 9$ as $(8 - 1) + 8 + (8 + 1)$ (line 3) in order to highlight the structure of the problem. These equivalent expressions help them make a general statement about adding an odd number of consecutive integers and the relationship of this sum to the middle summand. The Illustration also includes instances where students are reasoning about the sum of additive inverses (e.g., line 20). The students discuss this in a way that shows they recognize that numbers with opposite signs will sum to 0. For example, in lines 26 and 27, they realize they have to add all the integers from -15 to 17 . They don’t have to write anything down to realize immediately that the sum of all the integers from -15 to 15 will be 0.

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Student Materials

Suggested Use

Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. What does the word “consecutive” mean? Give three examples of sums of non-consecutive positive integers. How would lifting this restriction change the problem?
2. By the end of the Student Dialogue, what have the students shown? What is the original problem? What more do the students have to show? What is a possible next step?

Related Mathematics Tasks

1. Finish investigating this problem by taking the next step you identified in Student Discussion Question 3. Then keep going. Keep track of your conclusions for each step of your investigation. Even if a result doesn't seem important immediately, you may find that your observations will help you with further investigations.
2. Can all positive integers be expressed as the sum of two or more consecutive integers? If not, which ones can?
3. What is a way to determine how many odd factors a number has (and therefore how many different ways there are to write the number as a consecutive sum of positive integers) without writing out every possible factor?
4. Adding consecutive odd numbers starting with 1 seems to always result in a square number. For example, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$. Does this pattern continue? Use pictures and algebra to show how this pattern works.
5. Which numbers can be expressed in exactly *one* way as the sum of consecutive positive integers?

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Answer Key

Suggested Use

The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of the students in the dialogue engaging in the Standards for Mathematical Practice?

Refer to the Mathematical Overview for notes related to this question.

2. In this Illustration, we drop in on students as they work on this problem for the second day. What might the first day's conversation have looked like? How might you steer the first day's discussion so that students might have conversations like this one on the second day?

A large portion of the first day's conversation would likely involve making sense of the problem in one of two ways: either by finding sums to see what patterns emerge, or by starting with numbers and looking for all the different ways that numbers might be written as a sum of consecutive numbers. Students may find, for example, that every odd number may be written as the sum of two consecutive numbers, and that sums of three consecutive numbers result in numbers that are multiples of 3. Students may play with the sequence of triangular numbers by examining 1 , $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, etc. Students may find that they can make sums that add up to 3, 5, 6, 7, 9, 10, 11, and 12, but not 2, 4, or 8. To move the class in the direction you would like them to pursue on the second day, you might propose examining the sum of two consecutive numbers such as $6 + 7$ by rewriting the numbers in a form that shows a bit of the underlying structure: $6 + (6 + 1)$. In general the sum of two consecutive numbers can be written as $n + (n + 1)$, or $2n + 1$.

3. In what direction might the students take their further exploration of this problem?

The students say they want to see if there is a connection between factors and sums in general. They will likely want to try more numbers besides 33 to see how they behave, now that they have a method for generating likely consecutive sums. Students have seen a connection between odd factors and consecutive sums; another problem to explore might be to find a way to figure how many odd factors a number has. Students might also try to construct a logical argument for why a number with only even factors cannot be written as a consecutive sum. Many of these topics are addressed in the Mathematical Overview.

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4. In lines 29–33, the students look for another way to write 33 as a sum of consecutive numbers besides the ones they have already found. How do they know they only have to look for sums of 4 or 5 numbers?

They had already found 33 to be a sum of 2, 3, and 6 numbers. First, we know that it is impossible for any number to be written as a sum of 2 consecutive numbers in more than one way. 33 is $16 + 17$; no other 2 consecutive numbers can add to 33. $15 + 16$ (and any pair involving smaller numbers) is too small, and $17 + 18$ (and any pair involving larger numbers) is too large. Likewise, there is one unique set of 3 consecutive numbers and one unique set of 6 numbers that sum to 33. Next, a quick check confirms that adding 7 consecutive numbers won't work: $1 + 2 + \dots + 7 = 28$ and $2 + 3 + \dots + 8 = 35$, and higher combinations won't work. The smallest possible 8-number combination is $1 + 2 + \dots + 8 = 36$, which is already too large. So the only possibilities left are to try 4- and 5-number combinations. Providing justification for why the students only have to check sums of 4 and 5 numbers would have made the students' argument in the Student Dialogue stronger.

5. It is impossible for 33 to be written as a sum of 4 consecutive numbers. Without trying to calculate them, it is also possible to tell immediately that 195, 2911, 77, and 243 can't be written as sums of 4 consecutive numbers, either. Why not?

The sum of 4 consecutive numbers must be even. To be more specific, the sum of 4 consecutive numbers is always double an odd number (and any number > 6 that is twice an odd number can be written as the sum of 4 consecutive positive integers). We know the sum of 4 consecutive numbers must be even because of the 4 numbers, it is always the case that two of the numbers will be even and the other two will be odd. The sum of 2 even and 2 odd numbers is always even. Furthermore, we know the sum of four consecutive numbers is always double an odd number.

Consider the examples shown on the right, noticing the symmetry. We can also write the sum of 4 consecutive numbers with an algebraic expression:

$$\begin{array}{cc} \begin{array}{c} \overbrace{1+2+3+4}^5 \\ \underbrace{\hspace{1.5cm}}_5 \end{array} & \begin{array}{c} \overbrace{2+3+4+5}^7 \\ \underbrace{\hspace{1.5cm}}_7 \end{array} \end{array}$$

$(n - 1) + n + (n + 1) + (n + 2)$. (Of course, there are other ways to express this sum using algebra, but this choice highlights the sum of the middle two numbers.) The sum of the two middle numbers $n + (n + 1)$ is $2n + 1$, which we know to be odd. The sum of the outer two numbers $(n - 1) + (n + 2)$ is also $2n + 1$. So the sum of all four numbers is $2(2n + 1)$, or $4n + 2$, which is even.

6. In line 13, Sam asks “So if we can make *any* multiple of *any* odd number, can we also make any multiple of any even number?”

From the first part of his statement, we already know that we can make even multiples of odd numbers. We even have several ways to think about how to do this. For example, we can represent $2 \cdot 3$ as the sum of 3 numbers, with 2 being the middle number. From the response to Problem 5 above, we also know that we can think of $2 \cdot 3$ as the sum of 4 numbers, with $1 + 2$ being the middle two numbers. By coincidence, these two ways

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produce the same sum, $(0 +) 1 + 2 + 3$. But we have ways of thinking about even multiples of odd numbers.

All that remains now is to consider even multiples of even numbers. This case can be simplified even further. If we consider something like $4 \cdot 6$, or $8 \cdot 12$, each of these can be written as an odd multiple of an even number— $8 \cdot 3$ or $32 \cdot 3$ (respectively)—which are described above. So, really, the only case left to consider is the case when there are no odd factors at all... these are the powers of 2. By inspection, there does not appear to be any way to write 2, 4, or 8 as sums of consecutive integers. Perhaps this means that powers of 2 cannot be written as the sum of consecutive numbers (again the reader is invited to explore this question). For further discussion of these ideas, see the Mathematical Overview.

7. What evidence is there that the students will or will not get to the generalization that the task requests—finding a general rule for how many different ways a number can be written as a consecutive sum? What questions might you ask to guide them?

There is evidence of the students' mathematical thinking and their perseverance, so given some time, it could be easily argued that they will arrive at the general rule. To guide their thinking, if they don't experiment with this themselves, you could prompt them to consider which numbers have no odd factors or present them with some of the Related Mathematics Tasks at the end of this Illustration (particularly number 5). They could also be guided to a more explicit way to track their findings so far. Other thoughts are left for the readers to explore and discuss with colleagues.

8. What are some ways to modify the original problem to produce variations that may also provide some interesting opportunities to investigate numbers?

Some possibilities:

- Which numbers can be expressed exactly one way as sums of consecutive counting numbers?
- Which numbers can be expressed as products of consecutive counting numbers?
- Which numbers can be expressed as the sum of consecutive odd numbers?
- Which numbers can be expressed as sums of consecutive square numbers?

Possible Responses to Student Discussion Questions

1. What does the word “consecutive” mean? Give three examples of sums of non-consecutive positive integers. How would lifting this restriction change the problem?

Consecutive means that the numbers follow each other. Examples of sums of non-consecutive numbers are $1 + 100 = 101$, $6 + 6 + 6 + 6 = 24$, and $1 + 3 + 5 = 9$. Lifting this restriction would drastically change the problem because then any even number can be written as the sum of two odd numbers or two even numbers, so the only number that couldn't be written as the sum of two positive integers would be the number 1.

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2. By the end of the Student Dialogue, what have the students shown? What is the original problem? What more do the students have to show? What is a possible next step?

By the end of the Student Dialogue, students have shown that they can use any odd factor of a number to generate a set of consecutive numbers that sum to the number. For example, knowing that 5 is a factor of 30, the students have figured out that they can write 30 as the sum of 5 consecutive numbers, where the middle number is 6. They have seen that this method works, even if they have to “dip down” into non-positive integers. The original problem asks for a general rule for how many different ways a number can be written as a sum of consecutive positive integers. The students have yet to explicitly describe the connection between factors and consecutive sums. They also have not yet resolved Sam’s question (from line 13) about what they might be able to say (if anything) about making multiples of even numbers. As a next step, it probably makes sense for the students to continue to work with more specific examples (as they did for the number 33, which they summarize in lines 28–34) with both even and odd factors to see if they can come up with a general description or rule. Students might also consider exploring consecutive sums themselves to try to come up with an explanation for how every consecutive sum can be written as a product involving at least one odd factor.

Possible Responses to Related Mathematics Tasks

1. Finish investigating this problem by taking the next step you identified in Student Discussion Question 3. Then keep going. Keep track of your conclusions for each step of your investigation. Even if a result doesn’t seem important immediately, you may find that your observations will help you with further investigations.

The Mathematical Overview includes a proof that the number of ways a number can be written as a sum of consecutive numbers is one fewer than the number of odd factors it has. This is only one of many possible directions students might take their investigation. Students might also choose to examine the structure of sums with an even number of summands to see if they can find a way to express the general structure of those sums. See Teacher Reflection Question 5 for a brief discussion of sums of 4 consecutive numbers.

2. Can all positive integers be expressed as the sum of two or more consecutive integers? If not, which ones can?

Spoiler alert! No. The only numbers that cannot be expressed as the sum of two or more consecutive counting numbers are the powers of 2. This is the only set of numbers that has no odd factors.

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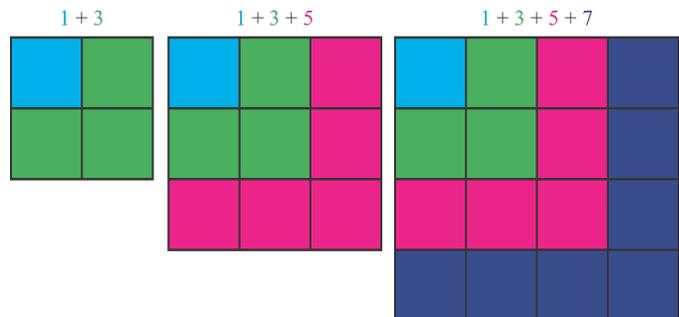
3. What is a way to determine how many odd factors a number has (and therefore how many different ways there are to write the number as a consecutive sum of positive integers) without writing out every possible factor?

First, as an example, find the number of odd factors of the number 300. Consider the prime factorization of $300 = 2^2 \cdot 3 \cdot 5^2$. Every factor of 300 is a combination of 2 (0, 1, or 2 of them), 3 (0 or 1 of them), and 5 (0, 1, or 2 of them). Total, there are $3 \cdot 2 \cdot 3 = 18$ total combinations, and so 300 has 18 factors. Out of these factors, the odd factors cannot include any multiple of 2. So there are $2 \cdot 3 = 6$ odd factors. To verify, we can find the odd factors: 1 ($=3^0 5^0$), 3 ($=3^1 5^0$), 5 ($=3^0 5^1$), 15 ($=3^1 5^1$), 25 ($=3^0 5^2$), and 75 ($=3^1 5^2$). So there are 5 ways to write 300 as the sum of consecutive positive integers.

In general, to find the number of odd factors of a number, write its prime factorization, ignore any power of 2 in the factorization, add 1 to each of the other exponents, and then multiply those augmented exponents together.

4. Adding consecutive odd numbers starting with 1 seems to always result in a square number. For example, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$. Does this pattern continue? Use pictures and algebra to show how this pattern works.

This picture is one way to demonstrate this pattern.



Here's one way to show this pattern with algebra:

$$\begin{aligned}
 2^2 &= 1 + (2 + 1) \\
 &= 2(1) + 2 \\
 3^2 &= 1 + (2 + 1) + (2(2) + 1) \\
 &= 3(1) + 2(1 + 2) \\
 4^2 &= 1 + (2 + 1) + (2(2) + 1) + (2(3) + 1) \\
 &= 4(1) + 2(1 + 2 + 3) \\
 5^2 &= 1 + (2 + 1) + (2(2) + 1) + (2(3) + 1) + (2(4) + 1) \\
 &= 5(1) + 2(1 + 2 + 3 + 4) \\
 n^2 &= 1 + (2 + 1) + (2(2) + 1) + (2(3) + 1) + \dots + (2(n - 1) + 1) \\
 &= n(1) + 2(1 + 2 + 3 + \dots + n - 1) \\
 &= n + 2(n(n - 1)/2) \\
 &= n + n(n - 1) \\
 &= n + n^2 - n \\
 &= n^2
 \end{aligned}$$

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5. Which numbers can be expressed in exactly *one* way as the sum of consecutive positive integers?

A number can be expressed in exactly one way as the sum of consecutive positive integers if it has exactly 2 odd factors. All prime numbers (except 2) can be written in only one way as the sum of two consecutive positive integers. Also, any odd prime multiplied by a power of 2 can only be written in one way as the sum of two or more consecutive positive integers. For example, 44 is $11 \cdot 4$ and has only 1 and 11 as odd factors. This means it can only be written one way as a sum of (2 or more) consecutive numbers (you figure out the string of numbers). All other numbers either have a single odd factor (these are the powers of 2, which all have a factor of 1) or more than 2 odd factors (any odd non-prime number must have at least 3 odd factors because if it only had 2, it would be prime, and if it has any factors of 2, it would be an even number).