# Extending Patterns with Exponents

**About Illustrations:** Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

**About the *Extending Patterns with Exponents* Illustration:** This Illustration's student dialogue shows the conversation among three students who are trying to find the value of expressions with negative integer exponents. The students come up with several possible definitions of negative exponents and evaluate which definition is most consistent with previously established rules for exponents.

**Highlighted Standard(s) for Mathematical Practice (MP)**
MP 1: Make sense of problems and persevere in solving them.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 6: Attend to precision.
MP 7: Look for and make use of structure.

**Target Grade Level:** Grades 8–9

**Target Content Domain:** Expressions and Equations

**Highlighted Standard(s) for Mathematical Content**
8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

**Math Topic Keywords:** rules for exponents, negative exponents



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# Extending Patterns with Exponents

## Mathematics Task

**Suggested Use**
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

How can we find the value of $2^{-3}$ and other expressions with negative exponents?

## Student Dialogue

**Suggested Use**
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

*Students in this dialogue have explored whole-number exponents with positive and negative integer bases and learned the rule for multiplying two exponential expressions with the same base. They are now investigating integer exponents with integer bases. They understand the properties of addition, subtraction, and parenthesis, and have number sense with fractions.*

(1) Chris: Well, $2^3$ is $2 \cdot 2 \cdot 2$, which is 8, so $2^{-3}$ should be -8, right?

(2) Lee: I'm not so sure. We know $-2^3$ is $-(2 \cdot 2 \cdot 2)$, which is -8, and if the negative is part of the base, like in $(-2)^3$, we get $-2 \cdot -2 \cdot -2$, which is also -8, but I don't know how to write out $2^{-3}$ as a multiplication problem.

(3) Chris: Hang on, we find $2^3$ and $-2^3$ or even $(-2)^3$ by multiplying the base three times. But what does $2^{-3}$ mean? How can you multiply something *negative three* times?

(4) Matei: You can't! There must be another way to think about exponents that will make sense even if the exponents are negative.

(5) Lee: Ok, let's list a few more positive powers of 2 and try to work backwards…

| $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ |
|---|---|---|---|---|---|---|---|---|
| 32 | 16 | 8 | 4 | 2 | | | | |

(6) Matei: Well, working backwards, we divide 32 by 2 to get 16, and we divide 16 by 2 to get 8, and so on… We divide by 2 each time the exponent goes down by 1.

(7) Lee: So, the next step should be 2 divided by 2, which is 1, so $2^0$ should be 1. And the next step divides by 2 again, to give us ½, so $2^{-1}$ should be ½, and so on…

| $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ |
|---|---|---|---|---|---|---|---|---|
| 32 | 16 | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

(8) Chris: I see what you guys did but I have another idea. I think $2^0$ should be 0 since you have zero copies of 2. And $2^{-1}$ should be -2 since that's the opposite of $2^1$, which is 2. That would make $2^{-2}$ equal to -4, and so on. What do you guys think?

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| $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ |
|---|---|---|---|---|---|---|---|---|
| 32 | 16 | 8 | 4 | 2 | 0 | -2 | -4 | -8 |

(9) Lee: Both methods follow a pattern. How do we decide which one makes the most sense for negative powers?

(10) Chris: *We* decide?! Isn't there a rule or something?

(11) Lee: Yes, I'm sure there's a rule for calculating negative exponents, but it has to make sense, so we can probably figure it out.

(12) Matei: Well, what else do we know about exponents? Um… we know how they are defined for positive whole-number powers; what else do we know?

(13) Chris: We know the rule for multiplying two exponential expressions with the same base. It's $a^n \cdot a^m = a^{n+m}$.

(14) Matei: So let's see which of our two patterns works for that rule. Let's try an example like $2^4 \cdot 2^{-1}$. Using the rules of exponents, we know $2^4 \cdot 2^{-1} = 2^{4+-1} = 2^{4-1} = 2^3$.

(15) Lee: Using the value $2^{-1} = \frac{1}{2}$ from our first pattern we get $2^4 \cdot 2^{-1} = 16 \cdot \frac{1}{2} = 8$. That's $2^3$. Using the conjecture that $2^{-1} = -2$ from our second pattern we get $2^4 \cdot 2^{-1} = 16 \cdot (-2) = -32$. That's not $2^3$, so $2^{-1} \neq -2$.

(16) Matei: So the first pattern makes more sense to follow because it gives us the same value for $2^4 \cdot 2^{-1}$ as we would expect using the rule of exponents $a^n \cdot a^m = a^{n+m}$.

(17) Chris: Hmm… the first pattern tells us to keep dividing by 2 each time we want to find a smaller power of 2. So, $2^0$ should be $2^1$ divided by 2 and that's 1.

(18) Lee: And $2^{-1}$ should be $2^0$ divided by 2, that's 1÷2, which is $\frac{1}{2}$. And $2^{-2}$ should be $2^{-1}$ divided by 2, so since $2^{-1} = \frac{1}{2}$, that means $2^{-2} = \frac{1}{4}$. And $2^{-3}$ should be...

(19) Matei: It's $\frac{1}{8}$!

(20) Lee and Chris: We did it!!

(21) Chris: But how can we find the value of other expressions with negative exponents?

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(22) Matei: Well, to decide on what $2^{-3}$ was, we used the exponent rule: $a^m \cdot a^n = a^{m+n}$. Can we use that to help us find the value of other expressions with negative integers?

(23) Lee: I think so. Let's say we want to find the value of $a^{-n}$. Well, we can know that $a^{-n} \cdot a^n = a^{-n+n}$. And $a^{-n+n} = a^0$ which we know is 1. So we have:

$$a^{-n} \cdot a^n = 1$$

$$a^{-n} = \frac{1}{a^n}$$

(24) Chris: That look right. But are we sure anything to the zero power is 1? How do we know it's always 1?

## Teacher Reflection Questions

**Suggested Use**
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. How is exponentiation initially defined? Does raising a number to a negative power make sense under this definition?

3. How do the students in this dialogue give meaning to negative powers?

4. What rule can be written to define raising a number to the zero power? Does this rule make sense for all numbers raised to the zero power?

5. How could you help students write a rule for raising a number to a negative power?

## Mathematical Overview

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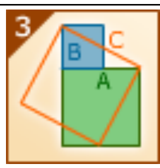
|---|
| **Suggested Use**<br>The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes. |

***Commentary on the Student Thinking***

| Mathematical Practice | Evidence |
|---|---|
| <br>Make sense of problems and persevere in solving them. | The students make sense of the problem by asking “How can you multiply something negative three times?” They each demonstrate making sense of the representation and what the negative exponent means. They try several approaches based on prior knowledge, test ideas, and construct new meanings. The students look for correspondence between the expressions with powers and the patterns in the table. |
| <br>Construct viable arguments and critique the reasoning of others. | The students’ attempts to come up with a sensible meaning for the symbol $2^{-3}$ illustrates MP 3 and 7. Students are defining what are, for them, new symbols—negative and zero exponents—and constructing a logical argument for their definition. And they are doing so in a way that preserves the behaviors they have come to depend on with positive integer exponents. They come up with two imaginable definitions based just on numeric pattern (lines 6–8). When they test their conjectures, they realize that one of their proposed definitions of negative exponents does not maintain the previously established rules of exponents, but their other definition does. In doing so, students asked questions such as, “What else do we know about exponents?” to help critique the two different mathematical ideas being offered. |
| <br>Look for and make use of structure. | Care about preserving established structure is the core of MP 7. While Matei/Lee work backwards to extend the “divide by 2” calculation, Chris suggests that $2^0 = 0$ and that negative powers of 2 will give the negative values of the positive powers of 2. Students eventually establish a definition for negative powers that feels completely consistent with previous rules and tells them how to evaluate such exponents. |
| <br>Attend to precision. | Students use words and algebra to communicate the patterns they identify with increasing precision. They attend to the values of the exponential expressions and make appropriate and specific use of known properties. Students define each subsequent value in the table as the previous number divided by the base, and use the equal sign meaningfully. The students use algebra to make a more precise rule for calculating negative powers (lines 21–23), based on what they have already said informally with words (lines 17–18). |

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***Commentary on the Mathematics***

Aside from exemplifying several mathematical practices, this dialogue also serves to make an important point: that some definitions in mathematics are based solely on their ability to extend previous rules and patterns. For example, exponentiation is defined as repeated multiplication of a base by itself however many times the power tells you to do so. However, this definition breaks down for zero, negative, and rational powers. To make meaning of such exponents, one must extend established rules and patterns as in the case of this dialogue. By extending a pattern and maintaining known rules of exponents like $a^n \cdot a^m = a^{n+m}$, the students are able to define negative exponents. A similar process of using established rules can be used to define rational exponents. (See the Illustration *Rational Exponents,* which explores the meaning of $64^{1/2}$ and $64^{1/3}$.)

Evidence of Content Standards

As students try to make sense of what a negative exponent might mean, they use properties of integer exponents they already know (8.EE.A.1). These properties include the definition of exponentiation for positive integer exponents (line 1) and the rule for multiplying two exponential expressions (line 13).

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## Student Materials

**Suggested Use**
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

***Student Discussion Questions***

1. How do you evaluate $a^n$ if both $a$ and $n$ are positive integers? Does multiplying $a$ by itself $n$ times make sense if $n$ is a negative integer?

2. What pattern do Lee and Matei, in the student dialogue, use to find the value for negative exponents?

3. Explain Chris's reasoning about what $2^0$ should mean.

4. What pattern does Chris use to find the values for negative exponents?

5. How do the students decide which of the two possible definitions for negative powers makes the most sense? Why do they use this method?

***Related Mathematics Tasks***

1. What rule can be written to define raising a number to the zero power? Is this rule always true? Provide two different explanations.

2. Prove that $(x^m)^n = x^{m \cdot n}$.

3. Students in the dialogue (line 2) noticed that $-2^3$ and $(-2)^3$ both equal -8. Does $-2^4 = (-2)^4$? How about $-5^3$ and $(-5)^3$ or $-5^4$ and $(-5)^4$? In what circumstances does $-a^n = (-a)^n$?

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## Answer Key

**Suggested Use**
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

***Possible Responses to Teacher Reflection Questions***

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. How is exponentiation initially defined? Does raising a number to a negative power make sense under this definition?

   When *a* and *n* are positive integers, $a^n$ is, by convention, the product of *n* copies of *a*. Based on this definition, negative exponents have no meaning since you can't have negative copies of a number.

3. How do the students in this dialogue give meaning to negative powers?

   Students begin by writing down powers of 2 they know and then look for a pattern that they can extend. They notice that the numbers are being divided by 2 as they move to the right (as the exponent decreases by 1). When Chris introduces a different pattern, students test one example to help verify which definition of a negative power preserves the rule of exponents: $a^n \cdot a^m = a^{n+m}$.

4. What rule can be written to define raising a number to the zero power? Does this rule make sense for all numbers raised to the zero power?

   The "rule" is a definition that can be written as $a^0 = 1$. This definition makes sense, as can be seen by writing out the evaluated expressions as done in the dialogue.

| $a^5$ | $a^4$ | $a^3$ | $a^2$ | $a^1$ | $a^0$ |
|---|---|---|---|---|---|
| $a•a•a•a•a$ | $a•a•a•a$ | $a•a•a$ | $a•a$ | $a$ | ? |

   When looking at the table above you can see that you are dividing by a each time you move to the right. When you divide a by a (as long as $a \neq 0$), you always get 1 so $a^0 = 1$. You can also use the exponent rule $a^n \cdot a^m = a^{n+m}$ to show that it makes sense to define $a^0 = 1$ if we want to preserve the rules for calculating with exponents. Here is why:

$$a^n \cdot a^m = a^{n+m}$$
$$a^n \cdot a^0 = a^{n+0}$$
$$a^n \cdot a^0 = a^n$$
$$\frac{a^n \cdot a^0}{a^n} = \frac{a^n}{a^n}$$
$$a^0 = 1$$

Alternatively, the exponent rule $\frac{a^n}{a^m} = a^{n-m}$ can also be used to show that $a^0 = 1$ is a sensible definition. Here is why:

$$\frac{a^n}{a^n} = a^{n-n}$$
$$1 = a^0$$

This logical chain of reasoning (MP 3) is not so much a proof that $a^0$ does equal 1 as an argument that it should equal 1. Because one cannot divide by 0, it doesn't actually work for $0^0$, but a limits argument with a approaching 0 from either direction does show that it makes sense to define $0^0$ as 1. Alas, that is not the only logical chain of reasoning that applies to $0^0$—it can also be argued plausibly that $0^0$ should be 0 (because $0^n$=0 for all other values of n)—and therefore, we are faced, once again, with a case not of logical inevitability or proof, but a deliberate choice between two sensible conclusions, a convention based on pragmatic considerations. It turns out that defining $0^0$ as 1 makes for greater consistency or greater utility, and so that is the convention.

5. How could you help students write a rule for raising a number to a negative power?

One way to help students would be to ask them to look for a pattern in the table for powers of 2. Have them compare a negative power with its positive counterpart. For example $2^2 = 4$ and $2^{-2} = \frac{1}{4}$. This will hopefully help students see the reciprocal relationship between positive and negative exponents and write the rule.

| | | | | | | |
|---|---|---|---|---|---|---|
| $2^3$ | $2^2$ | $2^1$ | $2^0$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ |
| 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

Another way to have students get to the rule is to encourage them to look at the table for powers of *a*. If they start with $a^0 = 1$ and divide by *a* each time they move to the right, they can then rewrite those divisions as multiplication by $\frac{1}{a}$. Once those expressions are simplified, they will see that $a^{-b} = \frac{1}{a^b}$.

| $a^3$ | $a^2$ | $a^1$ | $a^0$ | $a^{-1}$ | $a^{-2}$ | $a^{-3}$ |
|---|---|---|---|---|---|---|
| $a \cdot a \cdot a$ | $a \cdot a$ | $a$ | $1$ | $1 \div a =$ $1 \cdot \frac{1}{a} =$ $\frac{1}{a^1}$ | $1 \div a \div a =$ $1 \cdot \frac{1}{a} \cdot \frac{1}{a} =$ $\frac{1}{a^2}$ | $1 \div a \div a \div a =$ $1 \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} =$ $\frac{1}{a^3}$ |

***Possible Responses to Student Discussion Questions***

1. How do you evaluate $a^n$ if both *a* and *n* are positive integers? Does multiplying *a* by itself *n* times make sense if *n* is a negative integer?

   This definition does not make sense if *n* is negative, since you can't multiply a negative number of things. Instead $a^n$ can be evaluated as $\frac{1}{a^{-n}}$.

2. What pattern do Lee and Matei, in the student dialogue, use to find the value for negative exponents?

   Lee and Matei notice that as they move to the right through the evaluated powers of 2, they are dividing by 2 to get to the next number (dialogue lines 6–7). They extend this pattern to fill in the table for zero and negative exponents.

3. Explain Chris's reasoning about what $2^0$ should mean.

   Chris states that $2^0 = 0$ because you have zero copies of 2, which the student thinks should equal 0 (line 8).

4. What pattern does Chris use to find the values for negative exponents?

   Chris believes that the value for negative exponents will be the negative of the value for positive exponents (line 8). For example, Chris notices that $2^2 = 4$ and, therefore, reasons that $2^{-2} = -4$.

5. How do the students decide which of the two possible definitions for negative powers makes the most sense? Why do they use this method?

   The students test the values they get for negative powers of 2 against a rule of exponents they already know, specifically that $a^n \cdot a^m = a^{n+m}$. They first calculate an example, $2^4 \cdot 2^{-1}$, using the rule and then using the two different values they got for $2^{-1}$ from the patterns they had extended. This method of using an already existing rule for exponents is helpful (and necessary) since it allows students to check whether their definition for negative powers maintains previously established properties.

***Possible Responses to Related Mathematics Tasks***

1. What rule can be written to define raising a number to the zero power? Is this rule always true? Provide two different explanations.

   The rule that can be written is $a^0 = 1$. This rule can be seen by writing out the "evaluated" exponents as done in the dialogue.

| $a^5$ | $a^4$ | $a^3$ | $a^2$ | $a^1$ | $a^0$ |
|---|---|---|---|---|---|
| *a•a•a•a•a* | *a•a•a•a* | *a•a•a* | *a•a* | *a* | ? |

   When looking at the table above you can see that you are dividing by *a* each time you move to the right. When you divide *a* by *a* (as long as *a*≠0), you always get 1, so $a^0 = 1$. You can also use the exponent rule $a^n \cdot a^m = a^{n+m}$ to show that $a^0 = 1$. Here is how:

$$a^n \cdot a^m = a^{n+m}$$
$$a^n \cdot a^0 = a^{n+0}$$
$$a^n \cdot a^0 = a^n$$
$$\frac{a^n \cdot a^0}{a^n} = \frac{a^n}{a^n}$$
$$a^0 = 1$$

   Alternatively, the exponent rule $\frac{a^n}{a^m} = a^{n-m}$ can also be used to show that $a^0 = 1$ is a sensible definition. Here is why:

$$\frac{a^n}{a^n} = a^{n-n}$$
$$1 = a^0$$

   For more information about the case of $0^0$, see the notes on Teacher Reflection Question 5.

2. Prove that $(x^m)^n = x^{m \cdot n}$.

   Begin by expanding out $\left(x^m\right)^n$ as a product of *n* factors of $x^m$. Using the rule for multiplying exponents with like bases, we can rewrite the product using a sum of the exponents. Since this just becomes a sum of *n* terms of *m*, we can rewrite that expression as $m \cdot n$ to get $x^{m \cdot n}$.

$$(x^m)^n = x^{m \cdot n}$$

$$\overbrace{x^m \cdot x^m \cdot x^m \cdots x^m}^{\text{n terms}} = x^{m \cdot n}$$

$$x^{\overbrace{m+m+m+\cdots+m}^{\text{n terms}}} = x^{m \cdot n}$$

$$x^{m \cdot n} = x^{m \cdot n}$$

3. Students in the dialogue (line 2) noticed that $-2^3$ and $(-2)^3$ both equal -8. Does $-2^4 = (-2)^4$? How about $-5^3$ and $(-5)^3$ or $-5^4$ and $(-5)^4$? In what circumstances does $-a^n = (-a)^n$?

$$-5^3 = -125$$

$$(-5)^3 = -125$$

$$-5^4 = -625$$

$$(-5)^4 = 625$$

   $-a^n = (-a)^n$ when *n* is an odd integer. It does not hold true for even integer exponents since the left-hand side of the equation would be negative while the right-hand side would be positive.