Factoring a Degree Six Polynomial

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Factoring a Degree Six Polynomial* Illustration: This Illustration's student dialogue shows the conversation among three students who are exploring how to factor $x^6 - 1$ over the integers. They reason about different ways to use the structure of the expression to re-write the expression to facilitate factoring. They find multiple ways to do so and consider what that means about the expression and what can be factored and what cannot.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 3: Construct viable arguments and critique the reasoning of others.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 9–10

Target Content Domain: Seeing Structure in Expressions (Algebra Conceptual Category)

Highlighted Standard(s) for Mathematical Content

HSA-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*
HSA-SSE.B.3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

Math Topic Keywords: algebra, polynomials, factoring, substitution, chunking

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Factor $x^6 - 1$ over the integers.

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Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students know how to factor a difference of squares and a difference of cubes, and they know the factorization of $(x^3 + 1)$. They've also factored quadratic polynomials and higher degree polynomials that can be rewritten as quadratics using substitution, or what they call "chunking" (e.g. $x^4 + x^2 + 1$ as $(x^2)^2 + (x^2) + 1$).

(1) Matei: What does it mean to "factor over the integers"?

(2) Chris: It means that the factors have to have integer coefficients. Like, you *could* factor $x^2 - 2$ into $(x - \sqrt{2})(x + \sqrt{2})$, but that wouldn't count in this case. So $x^2 - 2$ doesn't factor *over the integers*.

(3) Lee: Here's what I'm thinking. If you factor $(x^2)^3 - 1$, you get $(x^2 - 1)(x^4 + x^2 + 1)$, which also factors into $(x + 1)(x - 1)(x^4 + x^2 + 1)$.

(4) Chris: Can $x^4 + x^2 + 1$ be factored more? What if we chunk it as $(x^2)^2 + (x^2) + 1$? Then we can have z equal x^2, and get $z^2 + z + 1$.

(5) Lee: I tried that, too. But $z^2 + z + 1$ doesn't factor, so neither does $x^4 + x^2 + 1$.

(6) Chris: Oh, ok. *[pauses]* Wait… Is that true?

(7) Lee: Is what true? You can't factor $z^2 + z + 1$.

(8) Chris: I know. But does that really mean that we can definitely say that we can't factor $x^4 + x^2 + 1$? Maybe there's a different way to factor expressions with x^4 that we haven't thought of. But my real question is, can we jump to the conclusion that $x^4 + x^2 + 1$ doesn't factor just because the chunked expression with z doesn't factor?

(9) Matei: Well, maybe we can come up with a different example. Let's see if we can start with something that we *know* factors and use chunking to turn it into something that doesn't.

[They sit and think for a few minutes, scribbling ideas down.]

(10) Chris: Here's one! What about the expression $x^2 - 1$? We know it factors. But then let z equal x^2. Then that expression turns into $z - 1$. Which doesn't factor. So there's your counterexample.

(11) Lee: Good one. So $x^4 + x^2 + 1$ *does* factor.

(12) Chris: Wait. I didn't say that. I just said that we can't say that it *doesn't* factor just because $z^2 + z + 1$ doesn't factor.

(13) Lee: Oh, right. So $x^4 + x^2 + 1$ *may* factor. Is that all we can say?

[They think for a minute.]

(14) Matei and Lee: *[together]* Look!

(15) Matei: *[to Lee]* You go first.

(16) Lee: Thanks. I found another way to factor $x^6 - 1$. If we write the expression as a difference of squares instead of a difference of cubes, we can factor it another way. Factoring $(x^3)^2 - 1$ gives $(x^3 + 1)(x^3 - 1)$, which is $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$. That means $x^4 + x^2 + 1$ must factor.

(17) Chris: And now we know it factors!

(18) Matei: I got the same result, but I did something different. I looked for a way to turn $x^4 + x^2 + 1$ into something I know is factorable, like $x^4 + 2x^2 + 1$.

(19) Chris: Um… Sure, that factors. That's $(x^2 + 1)^2$. But that's a different problem...

(20) Matei: But I can turn it into the same expression! Like this: $(x^4 + 2x^2 + 1) - x^2$.

(21) Lee: Ooh, neat! But where are you going with that?

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

2. Why is it important to specify that we are factoring "over the integers"?

3. Where is Matei going with "that" (line 21)?

4. In line 4, we see that $x^4 + x^2 + 1$ can be written as $z^2 + z + 1$ (when $z = x^2$). These expressions have the same underlying structure, so why is it that the first can be factored over the integers while the second one cannot?

5. In solving this problem, the students first jump to a false conclusion (line 5). Describe other common pitfalls involving algebraic equations and what seem like legal moves that often lead students to false conclusions.

6. If your students came to the same false conclusion as these did (line 5) but did not press forward to critique each other's reasoning, what would you do to intervene?

7. Precision in language plays an important role as the students go back and forth to clarify their ideas. Where do you see this in the dialogue?

8. What are some other ways of factoring $x^6 - 1$?

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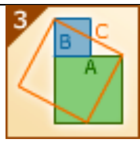
Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Construct viable arguments and critique the reasoning of others.	Lee presents an argument in line 5 ("$z^2 + z + 1$ doesn't factor, so neither does $x^4 + x^2 + 1$"). Chris questions in line 6 "Is that true?" In lines 8–10, the group answers that question with a counterexample. In lines 11–13, Lee makes a conjecture that Chris (again) counters by restating his own claim. Finally in lines 14–21, the students "make conjectures and build a logical progression of statements [to support] the truth of their conjecture" by finding alternative approaches to the problem.
 Look for and make use of structure.	In line 3, Chris changes x^6 to $(x^2)^3$ so that the original expression becomes a difference of cubes. Through chunking, in line 4 Chris changes the quartic expression into a quadratic in x^2 so the expression might look less complicated. Chris (line 10) again chunks an expression to provide a counterexample. In line 16, Lee interprets x^6 as $(x^3)^2$ in order to view $x^6 - 1$ as a difference of squares, which factors differently than the difference of cubes that Chris used. Finally, Matei also weighs in with structure in changing $x^4 + x^2 + 1$ to $(x^4 + 2x^2 + 1) - x^2$ so that the expression can be seen as a difference of squares. There is also a use of structure in line 17 when Chris claims to know the factors of $x^4 + x^2 + 1$. Making this statement requires Chris to compare $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$ to $(x + 1)(x - 1)(x^4 + x^2 + 1)$, recognize that they are both factored forms of $x^6 - 1$, see that $(x + 1)$ and $(x - 1)$ are equivalent, and thus conclude that the remaining portions must also be equivalent.

Commentary on the Mathematics

A basic assumption in this Illustration is that the system of polynomials derived throughout the students' work obeys the Fundamental Theorem of Arithmetic. That is, these polynomials can be factored into irreducible polynomials in only one way (the factors may be in any order). Thus, Lee's two factorizations

$$x^6 - 1 = (x-1)(x+1)(x^4 + x^2 + 1)$$

and

$$x^6 - 1 = (x-1)(x+1)(x^2 + x + 1)(x^2 - x + 1)$$

must be the same, leading to the identity

$$x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$$

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There is another approach to factoring $x^6 - 1$ over the integers that is foreshadowed in the CCSS description of MP 8:

> …Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, and $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead to the general formula for the sum of a geometric series…

This *Common Core* example asks students to reason about these calculations to arrive at the important identity (for positive integers *n*):

$$(x-1)(x^{n-1}+x^{n-2}+...+x^2+x+1)=x^n-1$$

Some students may choose to multiply everything by x and then by –1, noticing that all the terms in the expansion cancel to 0 except for the first and the last. Others may choose to multiply everything by $x - 1$, which, again, results in a "telescoping" sequence of terms leading to $x^n - 1$. Students in *CME Project* Algebra 2 have access to the factor theorem, which allows them to work this problem the other way. Since $x = 1$ is a solution to $x^n - 1 = 0$, then $x - 1$ is a factor of $x^n - 1$, and by dividing $x^n - 1$ by $x - 1$, they can arrive at the identity. Whichever way students get to the identity, it is one of the most useful in algebra. It's called the cyclotomic identity (*cyclotomy* means "circle dividing"). Gauss used it as a crucial piece of his characterization of those regular polygons that can be inscribed in a circle using only a straightedge and a compass.

This cyclotomic identity provides yet another factorization of the expression in the Student Dialogue:

$$x^6-1=(x-1)(x^5+x^4+x^3+x^2+x+1)$$

This form reveals the "sum of a geometric series" that is referenced in MP 8. Through this and other specific numeric examples, the repeated reasoning is intended to guide students to the formula

$$1+x+x^2+x^3+x^4+...+x^{n-2}+x^{n-1}=\frac{x^n-1}{x-1}$$

Chris, Lee, and Matei also scratch the surface of another interesting theorem. In their approach to this problem, they use substitution (aka *chunking*) to help them rewrite the expression into more helpful forms. For example, consider $x^{15} - 1$. This is a difference of cubes in disguise. That is:

$$x^{15}-1=(x^5)^3-1=♣^3-1$$

Since:

$$♣^3-1=(♣-1)(♣^2+♣+1)$$

Then

$$\begin{aligned}(x^5)^3-1&=(x^5-1)((x^5)^2+x^5+1)\\&=(x^5-1)(x^{10}+x^5+1)\end{aligned}$$

But, $x^{15} - 1$ is also a difference of fifth powers in disguise, which leads to a different factorization:

$$x^{15} - 1 = (x^3)^5 - 1 = ♣^5 - 1$$

Since (from the cyclotomic identity)

$$♠^5 - 1 = (♠ - 1)(♠^4 + ♠^3 + ♠^2 + ♠ + 1)$$

Then

$$x^{15} - 1 = (x^3)^5 - 1 = (x^3 - 1)((x^3)^4 + (x^3)^3 + (x^3)^2 + x^3 + 1)$$
$$x^{15} - 1 = (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1)$$

So we have another surprising identity:

$$(x^5 - 1)(x^{10} + x^5 + 1) = (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1)$$

This example hints at how the cyclotomic identity and chunking can be used to prove the following:

> **Theorem:** If *m* and *n* are integers and *m* is an integer factor of *n*, then $x^m - 1$ is a polynomial factor of $x^n - 1$.

The converse is also true: if $x^m - 1$ is a polynomial factor of $x^n - 1$, then *m* is an integer factor of *n*. These two results are the basis of the "polynomial factor game" in *CME Project* Precalculus.

Evidence of the Content Standards

The content standards come from the High School: Algebra (Seeing Structure in Expressions) domain. Students "interpret the structure of expressions" (HSA-SSE.A) and "write expressions in equivalent forms to solve problems" (HAS-SSE.B). To factor $x^6 - 1$, students rewrite the expression both as a difference of cubes, $(x^2)^3 - 1$, and as a difference of squares, $(x^3)^2 - 1$. Comparing the two expressions reveals new information, namely that the expression $x^4 + x^2 + 1$ factors over the integers and is the product $(x^2 - x + 1)(x^2 + x + 1)$. Students see that they can take advantage of rewriting (chunking) an expression like $x^4 + x^2 + 1$ into the expression $z^2 + z + 1$ (where $z = x^2$), but must also realize that two expressions with similar structures can behave differently.

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In line 10 of the student dialogue, Chris invents a counterexample. Why are the students looking for a counterexample? What does it show?

2. In lines 16 and 17, Lee and Chris claim that $x^4 + x^2 + 1$ factors and that they know the factors. How can they make that claim?

3. In line 3, Lee writes $x^6 – 1$ as $(x^2)^3 – 1$, while in line 16, he write the expression as $(x^3)^2 – 1$. Why does Lee rewrite the expression in these two different ways? Which way, in this case, turns out to be more helpful?

4. Where is Matei going with "that" (line 21)?

Related Mathematics Tasks

1. Factor $x^6 + 1$ over the integers. (Hint: Think $x^{12} – 1$)

2. Factor $x^4 + x^2 + 25$ over the integers.

3. How many irreducible factors does $x^n – 1$ have over the integers (as a function of n)? (Feel free to explore this problem using CAS technology.)

4. Play the Polynomial Factor Game (from *CME Project* Precalculus, modeled after the Factor Game from *CMP*).

   Two-player game:
   - Player 1 chooses any available polynomial on the board.
   - Player 2 identifies all polynomials on the board that are factors of that polynomial. If there are no factors for Player 2 to identify, then Player 1 loses a turn and no points are scored for the round.
   - Cross out (or otherwise remove) all the polynomials used in the turn from the board.
   - For each successive round, players alternate roles (picking the polynomial and finding the factors).

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Scoring:

- Player 1 scores points equal to the degree of the chosen polynomial.
- Player 2 scores points equal to the *sum* of the degrees of all of the factors identified.
- *Bonus*: If Player 1 finds a factor that Player 2 missed, Player 1 scores the points equal to the value of that factor.

Game board:

$x-1$	x^2-1	x^3-1	x^4-1	x^5-1
x^6-1	x^7-1	x^8-1	x^9-1	$x^{10}-1$
$x^{11}-1$	$x^{12}-1$	$x^{13}-1$	$x^{14}-1$	$x^{15}-1$
$x^{16}-1$	$x^{17}-1$	$x^{18}-1$	$x^{19}-1$	$x^{20}-1$

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

    Refer to the Mathematical Overview for notes related to this question.

2. Why is it important to specify that we are factoring "over the integers"?

    In this Illustration, we only consider polynomials with integer coefficients. Specifying this is important because this defines what we mean by "factorable" or "irreducible." For example, $x^2 + 1$ is irreducible (i.e., cannot be written as a product of factors of polynomials with integer coefficients) over the integers and is irreducible over the real numbers, but factors very nicely over the complex numbers as $(x + i)(x - i)$. To claim that $x^2 + 1$ "doesn't factor" is an imprecise statement unless it is specified that only polynomials with integer coefficients are allowed.

    Factoring "over the integers" also has implications for what we mean by "unique factorization." The polynomial $x^6 - 1$ has a unique factorization over the integers: $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$. All of the factors are irreducible (i.e., cannot themselves be written as a product of factors of polynomials with integer coefficients). The factorization can also be written $(x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1)$ but is still considered the same unique factorization because the order in which you write the factors does not matter. This is also the same factorization as $(-1)(-x - 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$ because over the integers, 1 and –1 are considered "units" and do not count as a separate "factor." (This is the same reason that for integers, 2 • 3 is a unique factorization for 6, even though 6 can also be written 1 • 2 • 3 or –2 • –3.)

3. Where is Matei going with "that" (line 21)?

    This is a different-than-typical version of completing the square. Instead of adding a constant, we add and subtract the middle term (in this case a quadratic term). The ultimate goal of both versions of completing the square is to purposefully transform the expression into a difference of squares. This is also a wonderful way to make the structure of an expression useable.

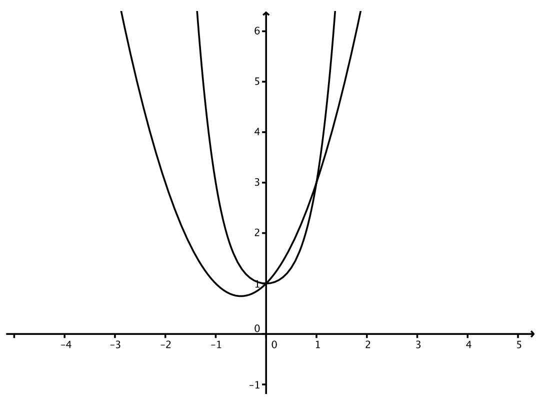
$$x^4 + x^2 + 1$$
$$x^4 + x^2 + 1 + (x^2 - x^2)$$

$$x^4 + 2x^2 + 1 - x^2$$
$$(x^2)^2 + 2(x^2) + 1 - x^2$$
$$(x^2 + 1)^2 - x^2$$
$$(x^2 + 1 + x)(x^2 + 1 - x)$$

This result matches what Lee got, but approaches it in a slightly different way.

4. In line 4, we see that $x^4 + x^2 + 1$ can be written as $z^2 + z + 1$ (when $z = x^2$). These expressions have the same underlying structure, so why is it that the first can be factored over the integers while the second one cannot?

These polynomials have a similar underlying structure in that $x^4 + x^2 + 1$ (call this $f(x)$) can be written as a composition of $g(z) = z^2 + z + 1$ and $h(x) = x^2$. In other words, $f(x) = g(h(x))$. But f and g are not the same function. Here they are, graphed on the same axes.



Which is which? (Hint: One of these functions is even.)

Because the expressions have the same underlying structure, we can apply Matei's completing-the-square approach in the same way to both of them. Here it is applied to $g(z)$:

$$z^2 + z + 1$$
$$z^2 + 2z + 1 - z$$
$$(z + 1)^2 - z$$

We can see that this approach does not help us factor $z^2 + z + 1$, but does lead to a difference of squares when $z = x^2$. So $x^4 + x^2 + 1$ factors over the integers whereas $z^2 + z + 1$ does not, though they have the same underlying structure.

5. In solving this problem, the students first jump to a false conclusion (line 5). Describe other common pitfalls involving algebraic equations and what seem like legal moves that often lead students to false conclusions.

Here are two fairly common situations that lead students to false conclusions. Both use the same apparently legal move—dividing by the variable.

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Example 1
$ax^2 = bx$
$ax = b$
$x = b/a$

Example 2
$ax = bx$
$a = b$

Without any context, the convention here is that both of these are equations to solve for some unknown number x. Each of the steps is correct provided that $x \neq 0$. But, in fact, $x = 0$ is a solution to both equations, and that solution "gets lost" if you divide both sides by x. A different way to approach these examples is to rewrite them so that one side is 0 and then solve using factoring and the zero product property.

In example 1:
$ax^2 = bx$
$ax^2 - bx = 0$
$x(ax - b) = 0$

Applying the zero product property: if $x(ax - b) = 0$, either $x = 0$ or $ax - b = 0$ (which leads to $x = \frac{b}{a}$). Thus the two solutions for Example 1 are $\{0, \frac{b}{a}\}$. See the Illustration *Making Sense of Unusual Results* for more about this idea.

Discuss other examples with colleagues to share ideas.

6. If your students came to the same false conclusion as these did (line 5) but did not press forward to critique each other's reasoning, what would you do to intervene?

   Discuss with colleagues to share ideas.

7. Precision in language plays an important role as the students go back and forth to clarify their ideas. Where do you see this in the dialogue?

   Right away in line 1, students clarify the meaning of "factoring over the integers" as being able to write an expression as the product of factors with integer coefficients. We also see students honing in on a more precise use of language in lines 10–13 when Chris and Lee make it clear that "I can't say it doesn't factor" is not the same as saying "it does factor." In this case, precise use of language helps to clarify the logic students are using to construct their argument. Though MP 6 (attend to precision) is not one of the highlighted Standards for Mathematical Practice in this Illustration, we see how careful communication supports the construction of mathematical arguments.

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8. What are some other ways of factoring $x^6 - 1$?

The students found

$$x^6 - 1 = (x + 1)(x - 1)(x^4 + x^2 + 1)$$
$$x^6 - 1 = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$$

There are other ways to write $x^6 - 1$ as a product of two or more factors, and we can get them by re-multiplying different combinations of factors back together. (This leads to another question: How many different ways *are* there to write $x^6 - 1$ as a product of two or more factors?)

Here's one rather interesting factorization of $x^6 - 1$:

$$x^6 - 1 = (x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)$$

Representing this product using an expansion box reveals an interesting pattern and even suggests a general method for factoring $x^n - 1$.

	x^5	x^4	x^3	x^2	x	1
x	x^6	x^5	x^4	x^3	x^2	x
-1	$-x^5$	$-x^4$	$-x^3$	$-x^2$	$-x$	-1

This factorization even has a special name: the general form is called the cyclotomic identity (see the Mathematical Overview for more on this).

Furthermore, these factorizations of $x^6 - 1$ so far only consider ways to factor this expression over the integers. If we lift that restriction, we can find other ways to write this product. We know from the Fundamental Theorem of Algebra that a degree-six polynomial must have six complex roots (not necessarily all distinct). So here's another way to factor $x^6 - 1$:

$$x^6 - 1 = (x+1)(x-1)\left(x + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(x + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(x - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(x - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

Possible Responses to Student Discussion Questions

1. In line 10 of the student dialogue, Chris invents a counterexample. Why are the students looking for a counterexample? What does it show?

The students are looking for a counterexample because they have made a claim, but they don't know if it is valid. Their claim (in line 5) is about whether $x^4 + x^2 + 1$ factors. The students use chunking, let $z = x^2$, and rewrite the expression as $z^2 + z + 1$, which they know is irreducible. Lee's claim is that since $z^2 + z + 1$ doesn't factor, neither does $x^4 + x^2 + 1$. The students figure that if they can find just one counterexample (i.e., a case in which a chunked expression written in z doesn't factor, but the same expression written in x does), they will know that the claim isn't valid. Chris invents the following as a counterexample: We can start with an expression that factors, like $x^2 - 1 = (x + 1)(x - 1)$, then let $z = x^2$ and rewrite the expression as $z - 1$, which is irreducible. The counterexample shows that their argument doesn't hold. It may still be true that $x^4 + x^2 + 1$ is irreducible (though they later show that it does factor), but they can't base their argument on the fact that $z^2 + z + 1$ is irreducible.

2. In lines 16 and 17, Lee and Chris claim that $x^4 + x^2 + 1$ factors and that they know the factors. How can they make that claim?

   In comparing $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$ to $(x + 1)(x - 1)(x^4 + x^2 + 1)$, Chris recognizes that they are both factored forms of $x^6 - 1$. Chris also sees that the factors $(x + 1)$ and $(x - 1)$ are the same, so the remaining portions $(x^4 + x^2 + 1)$ and $(x^2 - x + 1)(x^2 + x + 1)$ must also be equal. Underlying this reasoning is the valid assumption that the system of polynomials with integer coefficients has the unique factorization property (see the Commentary on the Mathematics and Teacher Reflection Question #2 for more).

3. In line 3, Lee writes $x^6 - 1$ as $(x^2)^3 - 1$, while in line 16, he write the expression as $(x^3)^2 - 1$. Why does Lee rewrite the expression in these two different ways? Which way, in this case, turns out to be more helpful?

   By rewriting the expression in different ways, Lee reveals more information about the expression $x^6 - 1$.

   The first shows a difference of cubes $(a^3 - b^3)$, which can be factored as $(a - b)(a^2 + ab + b^2)$. In this case, a is x^2 and b is 1. Using this method, we see that $x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1) = (x + 1)(x - 1)(x^4 + x^2 + 1)$.

   In the second, Lee rewrites the original as a difference of squares ($a^2 - b^2$, where a is x^3 and b is 1), which factors into two binomials, which are also easily factored (as a sum and difference of cubes). In this case, the second attempt at rewriting the expression proves to be more helpful because the result, $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$, contains only irreducible factors. Combining the two results provides new insight about how to factor $(x^4 + x^2 + 1)$.

   As an aside, if students are familiar with the identity, $x^3 - 1 = (x - 1)(x^2 + x + 1)$, then the identity for factoring $a^3 - b^3$ comes "for free" by writing the expression as $b^3\left(\left(\frac{a}{b}\right)^3 - 1\right)$.

4. Where is Matei going with "that" (line 21)?

   This is a different-than-typical version of completing the square. Instead of adding a constant, we add and subtract the middle term (in this case a quadratic term). The ultimate goal of both versions of completing the square is to purposefully transform the expression into a difference of squares. This is also a wonderful way to make the structure of an expression useable.

$$x^4 + x^2 + 1$$
$$x^4 + x^2 + 1 + (x^2 - x^2)$$
$$x^4 + 2x^2 + 1 - x^2$$
$$(x^2)^2 + 2(x^2) + 1 - x^2$$
$$(x^2 + 1)^2 - x^2$$
$$(x^2 + 1 + x)(x^2 + 1 - x)$$

This result matches what Lee got, but approaches it in a slightly different way.

Possible Responses to Related Mathematics Tasks

1. Factor $x^6 + 1$ over the integers. (Hint: Think $x^{12} - 1$)

We know that $(x^6 + 1)(x^6 - 1) = x^{12} - 1$. And we know that $x^6 - 1 = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$. So far, we can say

$$x^{12} - 1 = (x^6 + 1)(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$$

Factor $x^{12} - 1$ a different way, using difference of cubes:

$$x^{12} - 1 = (x^4)^3 - 1 = \clubsuit^3 - 1$$

Since

$$\clubsuit^3 - 1 = (\clubsuit - 1)(\clubsuit^2 + \clubsuit + 1)$$

Then

$$\begin{aligned}(x^4)^3 - 1 &= (x^4 - 1)((x^4)^2 + x^4 + 1)\\ &= (x^4 - 1)(x^8 + x^4 + 1)\\ &= (x^2 - 1)(x^2 + 1)(x^8 + x^4 + 1)\\ &= (x - 1)(x + 1)(x^2 + 1)(x^8 + x^4 + 1)\end{aligned}$$

To factor $x^8 + x^4 + 1$, use Matei's completing-the-square tip:

$$\begin{aligned}&= (x - 1)(x + 1)(x^2 + 1)(x^8 + 2x^4 + 1 - x^4)\\ &= (x - 1)(x + 1)(x^2 + 1)((x^4 + 1)^2 - x^4)\\ &= (x - 1)(x + 1)(x^2 + 1)(x^4 + 1 - x^2)(x^4 + 1 + x^2)\\ &= (x - 1)(x + 1)(x^2 + 1)(x^4 - x^2 + 1)(x^4 + x^2 + 1)\end{aligned}$$

One of these, $x^4 + x^2 + 1$, we already saw factored in the Student Dialogue. So,

$$x^{12} - 1 = (x - 1)(x + 1)(x^2 + 1)(x^4 - x^2 + 1)(x^2 - x + 1)(x^2 + x + 1)$$

We already saw that

$$x^{12} - 1 = (x^6 + 1)(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$$

So we find that

$$x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1)$$

2. Factor $x^4 + x^2 + 25$ over the integers.

We can use Matei's completing-the-square and difference-of-squares combination:

$$x^4 + x^2 + 25 = x^4 + 10x^2 + 25 - 9x^2$$

$$= (x^2 + 5)^2 - 9x^2$$
$$= (x^2 + 5 - 3x)(x^2 + 5 + 3x)$$
$$= (x^2 - 3x + 5)(x^2 + 3x + 5)$$

3. How many irreducible factors does $x^n - 1$ have over the integers (as a function of n)? (Feel free to explore this problem using CAS technology.)

Students can explore examples:

n	Irreducible factors	Number of factors
2	$x^2 - 1 = (x - 1)(x + 1)$	2
3	$x^3 - 1 = (x - 1)(x^2 + x + 1)$	2
4	$x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$	3
5	$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$	2
6	$x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$	4
7	$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$	2
etc.		

The number of irreducible factors of $x^n - 1$ over the integers is the number of divisors of n.

4. Play the Polynomial Factor Game (from *CME Project* Precalculus, modeled after the Factor Game from *CMP*).

Two-player game:
- Player 1 chooses any available polynomial on the board.
- Player 2 identifies all polynomials on the board that are factors of that polynomial. If there are no factors for Player 2 to identify, then Player 1 loses a turn and no points are scored for the round.
- Cross out (or otherwise remove) all the polynomials used in the turn from the board.
- For each successive round, players alternate roles (picking the polynomial and finding the factors).

Scoring:
- Player 1 scores points equal to the degree of the chosen polynomial.
- Player 2 scores points equal to the *sum* of the degrees of all of the factors identified.
- *Bonus*: If Player 1 finds a factor that Player 2 missed, Player 1 scores the points equal to the value of that factor.

Game board:

$x - 1$	$x^2 - 1$	$x^3 - 1$	$x^4 - 1$	$x^5 - 1$
$x^6 - 1$	$x^7 - 1$	$x^8 - 1$	$x^9 - 1$	$x^{10} - 1$
$x^{11} - 1$	$x^{12} - 1$	$x^{13} - 1$	$x^{14} - 1$	$x^{15} - 1$
$x^{16} - 1$	$x^{17} - 1$	$x^{18} - 1$	$x^{19} - 1$	$x^{20} - 1$

Factoring a Degree Six Polynomial

Students can come up with strategies for success in the Polynomial Factor Game. For example, Player 1 will find that choosing $x^{12} - 1$ is a poor first move, as they will get 12 points, while Player 2 can get 16 points (for $x - 1$, $x^2 - 1$, $x^3 - 1$, $x^4 - 1$, $x^6 - 1$) on that turn. By reasoning using a combination of difference of squares and difference of cubes, students will find in general that $x^n - 1$ will divide $x^m - 1$ if n divides m.