Finding Triangle Vertices

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Finding Triangle Vertices* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the third vertex of triangle given two vertices and the area of the triangle. After trying several approaches, the students realize that there are an infinite number of possible points that could be the third vertex.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 6: Attend to precision.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 6–7

Target Content Domain: Geometry

Highlighted Standard(s) for Mathematical Content

6.G.A.3	Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
6.G.A.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
7.G.B.6	Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Math Topic Keywords: coordinate plane, area of a triangle

This material is based on work supported by the National Science Foundation under Grant No. DRL-1119163. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Finding Triangle Vertices

Mathematics Task

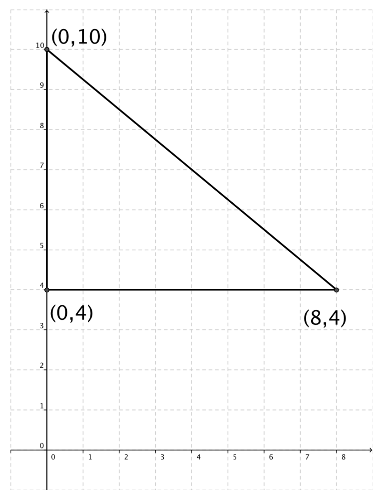
Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Two vertices of a triangle are located at (0,4) and (0,10). The area of the triangle is 12 square units. Where can the third vertex be located?

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students in this dialogue have already learned how to find the area of a triangle given the base and height. Today's task requires these students to apply that knowledge "inside out": What can be determined about a triangle when one is given the area and base? Their teacher assigned this task to allow the students to explore and extend their understanding of triangle area and its relationship to base and height and to engage in the MPs involved in problem solving and generalizing.



(1) Sam: So let's pick a point for the third vertex and see if the triangle has an area of 12 square units. Let's try (8,4).

(2) Dana: Ok, so, um, to figure out the area of the triangle, we take half the base times the height.

(3) Sam: So if we use (8,4), then the base is 8. See? I'm just counting here. *[points along the horizontal leg]* And the height is 6. Right?

(4) Anita: That's right. Or a base of 6 *[points along the vertical leg]* and a height of 8 *[points along the horizontal leg]*, depending how you look at it. Like if you turn the picture around.

(5) Sam: Great, so let's plug our values into the formula. *[writes down:* $A = \left(\frac{1}{2}\right)(8)(6)$ *and pauses]* Eight times six is 48, and half of that is 24. Hmm, that didn't work. I guess we need to pick a different point.

(6) Dana: Instead of randomly picking a point, how about we work backwards from what we know.

(7) Sam: Okay, so we got 24 with our first try. Working backwards… let's see, we want an area that is half that size. Half of (8,4) is (4,2). I bet (4,2) will work!

(8) Anita: No, look where the point (4,2) is. *[points at (4,2) in the picture]* That triangle doesn't look like half. Plus, what would the base and height be?

(9) Dana Guys, that might work. I don't know. But that's not what I meant by working backwards! I meant let's work backwards using the formula. We know our

triangle needs an area of 12, and we can say the base is equal to 6 units. So, we can figure out what the height must be. Let's see… We get 12 equals… *[writes:* $12=\left(\frac{1}{2}\right)(6)(h)$ *]* That's like multiplying *h* by 3, so *h* must be 4, because 3 times 4 is 12.

(10) Sam: Okay. I see. So we want to go 4 away from (0,4), not 8 away like we first tried with our guess of (8,4). So… (4,4)!

(11) Dana: That works… *[writes:* $A=\left(\frac{1}{2}\right)(4)(6)=12$ *]*

(12) Anita: *[Measures and marks 4 units on a piece of string and then starts marking other points on the paper that are 4 away from the point (0,4).]*

(13) Sam: What are you doing, Anita?

(14) Anita: I think there are other points that work. You and Dana said it has to be 4 away from (0,4) so I'm measuring to find where else is 4 away. Look! The points that are 4 away form a circle!

(15) Dana: Oh! Yes! Cool! But that's not what we're doing. We said the height of the triangle is 4 … those other points you're drawing are not heights of the triangle.

(16) Anita: I know. I just thought it was interesting! So what *do* we do?

(17) Sam: We have one vertex that works. Let's each see if we can figure out a different vertex that makes a height of 4 and compare our answers to check.
[students work individually for a minute]

(18) Sam: I got (4,7).

(19) Dana: How did you get that? I got (4,10).

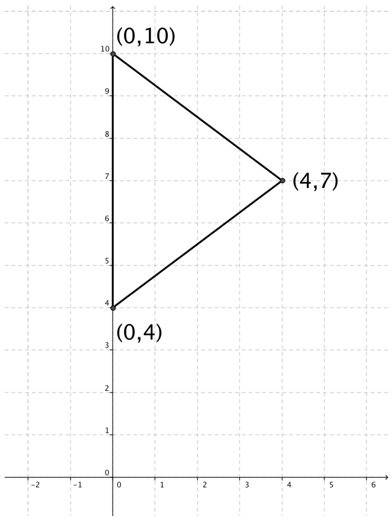
(20) Anita: I got (4,10), too. What did you do, Sam?

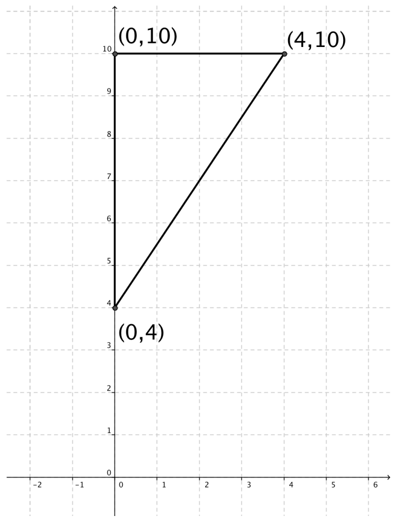
(21) Sam: It's easy! I started here in the middle of the base *[points to (0,7) on the drawing]* and counted over 4 units to get (4,7).

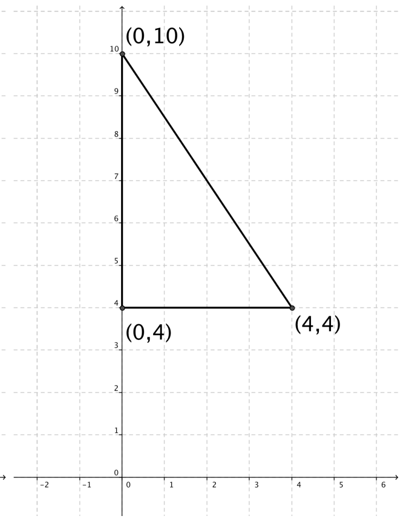
(22) Dana: But you don't have to start there. I started at (0,10) and counted 4 to get (4,10).

(23) Anita: And we already know you can start at (0,4), and move right 4 to get (4,4).

Finding Triangle Vertices







(24) Dana: Well, all our triangles have areas of 12 so I guess they are all right.

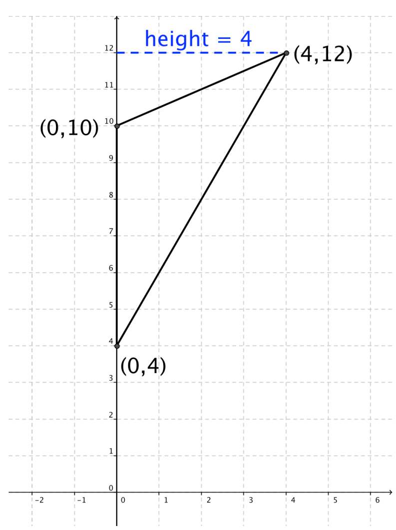
(25) Anita: Wait! I think we could count over 4 from *any* point on the base. So our third vertex could be anywhere between (4,4) and (4,10).

(26) Dana: Oh! That's kind of like what you were doing before with the string, Anita! But now it's distance from the *line*, not distance from a *point*!

(27) Sam: Hmmm. Maybe it doesn't have to be between (4,4) and (4,10). The 4 keeps showing up in the *x*-coordinate. Could *any* point with an *x*-coordinate of 4 work? What about the point (4,12)?

(28) Dana: Well, if you plot the point (4,12), what's the height? It doesn't look like it's 4 units from the base.

(29) Anita: Sure it is! You can extend the base of the triangle.
[extends the base to line x = 0]



See? The height is still 4, so the point (4,12) works. You can slide the point up and down along the line defined by $x = 4$, and it will always be 4 units away from the base of the triangle.

(30) Sam: Wow! So that means an infinite number of points—*anywhere along that line*—could be the third vertex! I wonder if any *other* points would work!

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see in the dialogue of students engaging in the Standards for Mathematical Practice?

2. Have the students in the dialogue found all possible coordinates for the third vertex of the triangle? Why or why not?

3. What actions or questions would you use to get the students in the dialogue to extend their reasoning and find the second set of possible vertices?

4. Suppose that the students in the dialogue were working in three dimensions. Given that two vertices are at (0,4,0) and (0,10,0), where could the third vertex of the triangle be located?

5. A square pyramid has the following vertices on the *xy* plane: (-2,2), (-2,6), (2,6), and (2,2). Where is the fifth point of the pyramid located so that the volume equals 48 cubic units?

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	**Evidence**
 Make sense of problems and persevere in solving them.	MP 1 says "Mathematically proficient students start by…looking for entry points…and plan a solution pathway rather than simply jumping into a solution attempt." It might, at first, seem that Sam's dive-right-in approach (line 1 of the dialogue) is quite the opposite. But the student seems to know that the strategy chosen is "just pick[ing] a point" and experimenting concretely with what it takes to "see if the triangle has an area of 12 square units." This is a reasonable first step in making sense of the problem, and all three students are involved in that experiment. Dana, by line 6, is ready for a different approach. In line 9, Dana suggests using the area formula not to *check* a point they pick at random, but to find the height of the triangle. At this stage, Dana, too, is making sense of the problem. From the little said, it seems Dana has only a partial plan for solving the problem, because the method used calculates a height, not a vertex. However, by saying that they should work backwards, Dana seems to see how their previous experiment (starting with a point and finding the area) and their new experiment (starting with the givens), might get them close to finding a point. These students work toward a solution relatively efficiently, and so their "perseverance" is not tested as much as their ability to make sense of the problem and plan and follow through on a solution method.
 Look for and express regularity in repeated reasoning.	Students assume that there is a unique answer, and are surprised when they find different points for the third vertex. When that happens, it's natural for students to ask "how did you get that" and explain their process. The three solutions that the students found were not the result of "repeated reasoning" by any one of them, but Anita and Sam both notice and express regularity in what they have discovered (MP 8). Anita (line 25) describes regularity in their *process* ("count over 4 from…the base"), and Sam describes regularity in their *result*, the x-coordinate of the third vertex, which leads him to (line 27) experiment outside the box (literally), which Anita (line 29) then states as a conjecture, adding the notion of "sliding," which would include even those points between the lattice lines of the grid. At the end, Sam pushes that conjecture even further, looking for still other possible places for the third vertex.

 Attend to precision.	The dialogue also shows students working out precision of meaning. Anita, first in line 4 and later in line 29, makes clearer the extent of what “base” means. As a sixth grader, Anita still wants “base” to carry the sense of “bottom” that it has in casual use (like the base of a lamp), so Anita needs to “turn the picture around” to justify calling the base 6 and the height 8, but the student *is* recognizing and making explicit that there is a choice: either of the two perpendicular sides can be considered the base or the height. The hypotenuse could also be used as the base, however it would then require extra work to find the height and, in non-right triangles, we must always face the issue of how to compute the height perpendicular to the side we’ve chosen to be the base. In line 29, Anita shows how a segment representing the altitude may lie outside the triangle. Students often struggle with this because the examples they are used to so often illustrate height as a dotted line inside the triangle.

Commentary on the Mathematics

This task is neither the straightforward “find the area of a triangle with base b and height h” nor the common “backward” version “given the area and base (or height), find the height (or base).” Such questions have unique right answers; there’s little call to “think beyond the answer.” By contrast, problems with infinitely many correct (and incorrect) answers require genuine engagement in mathematical practice.

This dialogue highlights the idea of non-unique solutions: in this case, the fact that infinitely many triangles satisfy the givens (MP 1), which include one side of the triangle (in this case specified as the two endpoints of that side) and the area. The third vertex, the goal of the problem, must lie on a line parallel to the base so that a constant height (and, therefore, a constant area) is ensured. In line 30, Sam wonders if any other points would work. Following this up may lead to the discovery of another parallel line, defined by $x = -4$, that can contain the third vertex. This second line, a reflection of their first solution line across the triangle’s base, is an excellent example of how reflections preserve distance between objects.

Because the problem has infinitely many solutions, it can be extended by asking what additional constraints might limit the solutions to a finite number. Can it be limited to exactly 10 solutions? 2? 1? Or, instead of focusing on the number of solutions, we might focus on the nature of the constraint. What effect would constraining the perimeter have? The fact that it *has* an effect shows its non-relationship with area. Alternatively, what effect on perimeter do the varying solutions have? As the third vertex moves farther up or down the lines defined by $x = 4$ and $x = -4$, the area stays constant but the perimeter varies. Students can also be asked to consider an optimization problem by looking for the third vertex that would produce a triangle with the specified area but would minimize the perimeter. In sixth grade, even a question like “what is the greatest possible perimeter?” might be a surprise.

This dialogue also highlights a common student confusion about what the height of a triangle is. We see Anita pushing the students to remember that there is more than one possible base to a triangle, though Anita wants the base to be on the “bottom” and so references “turning” the picture. Anita later (line 14) momentarily loses track of what the number 4 is meant to represent and starts measuring distance *from a point* (forming a circle around one of the vertices), rather than distance *from a line* as a height represents.

Finding Triangle Vertices

<u>Evidence of the Content Standards</u>

Students in this dialogue are asked to determine where a triangle's third vertex can be given a set of conditions. This directly relates to the drawing polygons in the coordinate plane standard (6.G.A.3). Furthermore the problem involves area of a triangle (7.G.B.6) which is constrained to a 12 square units. The formula for area of a triangle is used (6.G.A.1) both in the typical "solve for area" way (see line 5) but in an unexpected way, working backwards from area to solve for the triangle's height (see line 9).

Finding Triangle Vertices

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In line 4 of the dialogue, Anita said that there were *two* sides of the triangle that could be called the height. Is Anita right? Can you have a triangle where all *three* sides could be called the height? Why or why not?

2. In line 5, Sam sees that the first point they tried (8,4) did not work because that made the area too large. Suppose, instead, that Sam had decided to guess and check another, for instance, (10,4) or (6,4). Without actually calculating the area of either of the two resulting triangles, explain why one point is a more sensible guess than the other.

3. After the height of the triangle was calculated, the students found three different coordinates for the third vertex. Which was right and why?

4. After choosing one side of a triangle as a base, the altitude (height) of the triangle is defined as the distance from *the vertex opposite that base to the line that contains the base*. The distance from a point to a line is defined as the shortest path from the point to the line, and the shortest path is always *perpendicular* to the line. Use only those two definitions to defend or reject Anita's claim that the height of a triangle can be measured *outside* of the triangle.

5. How many possible points do the students find for the third vertex? Where are all these points located? What makes this location "work"?

6. Can the point (4,5.5) be a third vertex? Explain why or why not? What about (4,10.25)? Or $(4,\pi)$?

7. Sam (line 30) wonders if there are still other points that would work. Are there? Make a clear argument for your answer.

Finding Triangle Vertices

Related Mathematics Tasks

1. *(For sixth- and seventh-grade students.)* Two vertices of a triangle are located at (-1,-5) and (7,-5). The area of the triangle is 24 square units. Where can the third vertex be?

2. *(For seventh- and eighth-grade students.)* A point is located 5 units away from the point (7,8). Where can the point be located?

3. *(For high school students.)* Two vertices of a triangle are located at (4,1) and (4,5). The perimeter of the triangle is 12 units. Where can the third vertex be located? What tools might help you explore this situation?

Finding Triangle Vertices

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. Have the students found all possible coordinates for the third vertex of the triangle? Why or why not?

   No. They've found the right idea—any point 4 units away from the base—but haven't seen all of its implications. The vertex could also lie on the "other side" of the base, the line defined by $x = -4$, which is the reflection of the line defined by $x = 4$ across the base of the triangle.

3. What actions or questions would you use to get the students in the dialogue to extend their reasoning and find the second set of possible vertices?

   Once students have found that the third vertex could lie anywhere along the line defined by $x = 4$, they can be encouraged to cover up the right side of the coordinate plane and ask themselves where else a point 4 units away from the base could lie.

4. Suppose that the students in the dialogue were working in three dimensions. Given that two vertices are at (0,4,0) and (0,10,0), where could the third vertex of the triangle be located?

   The third vertex would lie on a cylinder with a radius of 4 units about the y-axis (the extended base of the two given vertices). For any point on the y-axis, the distance of a point (x,y,z) from the y-axis is $\sqrt{x^2+z^2}$, so the equation $\sqrt{x^2+z^2} = 4$ specifies all points whose distance from the y-axis is 4. That equation can be rewritten as $x^2+z^2=16$, a *circle*! Except that, because we're in three dimensions, that circle is reproduced for every value of y, so it becomes the formula for a cylinder! It can also be written as $\left(\frac{x}{4}\right)^2+\left(\frac{z}{4}\right)^2=1$.

5. A square pyramid has the following vertices on the *xy* plane: (-2,2), (-2,6), (2,6), and (2,2). Where is the fifth point of the pyramid located so that the volume equals 48 cubic units?

   The fifth point would lie on a plane parallel to the *xy* plane, passing either through point (0,0,9) or through point (0,0,-9). The equations of these planes are $z = 9$ and $z = -9$.

Possible Responses to Student Discussion Questions

1. In line 4 of the dialogue, Anita said that there were *two* sides of the triangle that could be called the height. Is Anita right? Can you have a triangle where all *three* sides could be called the height? Why or why not?

   Anita's right. The *base* of a triangle must always be one of its sides, but any side could be chosen as the base. It's different for height. Height is always measured perpendicular to the base you choose, so only when two sides are perpendicular to each other—that is, only when you have a right triangle—can a side be the "height." If two sides are at right angles to each other—that is, if we have a right triangle—then either can be chosen as base, and the other is the same length as the triangle's height (altitude). It's not possible to have a triangle on a plane where all three choices of base have another side perpendicular to them, so never can more than two sides be called the height.

2. In line 5, Sam sees that the first point they tried (8,4) did not work because that made the area too large. Suppose, instead, that Sam had decided to guess and check another, for instance, (10,4) or (6,4). Without actually calculating the area of either of the two resulting triangles, explain why one point is a more sensible guess than the other.

   Without calculating, we don't know whether either point is correct, but we can see just by looking that the point (10,4) produces a triangle of greater area so (6,4) is the more sensible guess.

3. After the height of the triangle was calculated, the students found three different coordinates for the third vertex. Which was right and why?

   All three possible vertices that the students found were correct. Each was found by counting over 4 units (the height) and produced a triangle with an area of 12 square units.

4. After choosing one side of a triangle as a base, the altitude (height) of the triangle is defined as the distance from *the vertex opposite that base to the line that contains the base*. The distance from a point to a line is defined as the shortest path from the point to the line, and the shortest path is always *perpendicular* to the line. Use only those two definitions to defend or reject Anita's claim that the height of a triangle can be measured *outside* of the triangle.

   The picture Anita draws illustrates the proof. Anita chooses the segment (0,4) to (0,10) as the base. The *y*-axis is a line through that base. The student measures perpendicular to that line, from that line to the vertex (4,12) opposite the base. So that measurement is the height of the triangle and is entirely outside the triangle.

5. How many possible points do the students find for the third vertex? Where are all these points located? What makes this location "work"?

   The students find that infinitely many points, every point on the line defined by $x = 4$, could be the third vertex. This line works because all of its points are 4 units away from the line that contains the base of the triangle. This height produces a triangle with an area of 12 square units as demanded by the problem.

6. Can the point (4,5.5) be a third vertex? Explain why or why not? What about (4,10.25)? Or $(4,\pi)$?

   Yes. All those points lie on the line defined by $x = 4$ and are, therefore, 4 units from the triangle's base.

7. Sam (line 30) wonders if there are still other points that would work. Are there? Make a clear argument for your answer.

   All of the points on the line defined by $x = -4$ are also a distance of exactly 4 from the base of the triangle, so any of them can be the third vertex of a triangle with an area of 12.

Possible Responses to Related Mathematics Tasks

1. *(For sixth- and seventh-grade students.)* Two vertices of a triangle are located at (-1,-5) and (7,-5). The area of the triangle is 24 square units. Where can the third vertex be?

   The third vertex can be located on the lines defined by the equations $y = 1$ and $y = -11$. The height of the triangle must be 6 units.

2. *(For seventh- and eighth-grade students.)* A point is located 5 units away from the point (7,8). Where can the point be located?

   The point is located on the circle with a radius of 5 and a center at (7,8). Possible points include (7,13), (7,3), (2,8) and (12, 8).

3. *(For high school students.)* Two vertices of a triangle are located at (4,1) and (4,5). The perimeter of the triangle is 12 units. Where can the third vertex be located? What tools might help you explore this situation?

   The two given vertices are 4 units apart. The other two sides of the triangle must account for another 8 units exactly to make the perimeter 12 units. An 8-unit piece of string "tacked" to the two known vertices might be a useful tool for exploring and making conjectures, as might geometry software. A little play suggests that the third vertex can be (almost) any point on what looks like an ellipse. In fact, the curve is an ellipse, specifically the one given by:

$$\frac{(x-4)^2}{(\sqrt{12})^2}+\frac{(y-3)^2}{4^2}=1$$

It is "*almost* any point" because (4,-1) and (4,7) are collinear with (4,1) and (4,5) (that is, they lie on the extended base) and, therefore, do not produce triangles. "Nice" points that students might find include (1,5), (1,1), (7,5) and (7,1) since those produce right triangles with the ellipse's foci.