Integer Combinations—Postage Stamps Problem (MS Version)

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Integer Combinations—Postage Stamps Problem (MS Version)* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the amounts of postage that are impossible to make using only five-cent and seven-cent stamps. While going through each positive whole number and determining if that amount is possible, students find that above a certain value all postage amounts are possible and they explore why.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 4–7

Target Content Domain: Operations and Algebraic Thinking, Expressions and Equations

Highlighted Standard(s) for Mathematical Content

4.OA.B.4	Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.
4.OA.C.5	Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*
5.OA.A.2	Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, and then multiply by 2" as $2 \times (8 + 7)$. Recognize that $3 \times (18932 + 921)$ is three times as large as $18932 + 921$, without having to calculate the indicated sum or product.*
6.EE.A.2	Write, read, and evaluate expressions in which letters stand for numbers.

Math Topic Keywords: factors, multiples, relatively prime numbers

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Integer Combinations—Postage Stamps Problem (MS)

Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Suppose the post office only sold five-cent stamps and seven-cent stamps. Some amounts of postage can be made with just those two kinds of stamps. For example, 1 five-cent and 2 seven-cent stamps make 19 cents in postage, and 2 five-cent stamps makes 10 cents in postage. Which amounts of postage is it impossible to make using only five-cent and seven-cent stamps?

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students in this dialogue have learned about factorization and multiples. They are now exploring what numbers can be produced by adding only two types of numbers.

(1) Sam: Well, let's see. I can make postage that is 5 cents, 10 cents, 15 cents, and so on using only five-cent stamps. And I can make postage that is 7 cents, 14 cents, 21 cents, and so on using seven-cent stamps.

(2) Dana: So, it looks like all multiples of 5 and 7 can be made.

(3) Anita: Yes, but so can combinations of different stamps. Like the problem said, you can have 1 five-cent stamp and 2 seven-cent stamps to make 19 cents.

(4) Dana: You're right. Why don't we make a table to keep track of everything? How about a table like this: *[draws the following]*

Total Postage	How?

We can already start filling out some of this. We know we can't make any postage smaller than 5 since the five-cent stamp is the smallest stamp we have.

Total Postage	How?
1	Impossible
2	Impossible
3	Impossible
4	Impossible
5	1 five-cent stamp

(5) Sam: Ok, so let's keep filling in some of this table and see what postage we can and can't make. 6 is impossible since the next smallest postage we can make is 10, by using 2 five-cent stamps.

(6) Dana: Well, that would mean everything from 6 through 9 is impossible. Oh wait, we could have 1 seven-cent stamp. And then we can have 10 using 2 five-cent stamps. 11 we can't make since 5 + 5 = 10 and 5 + 7 = 12. 12 cents of postage we can have.

Total Postage	How?
6	Impossible
7	1 seven-cent stamp
8	Impossible
9	Impossible
10	5 + 5
11	Impossible
12	5 + 7

(7) Anita: Well, if we can make 12, we can make any other postage that ends with a "2" since we just keep adding 10 using 2 five-cent stamps.

(8) Dana: You're right. But let's go in order before we jump around again. So 13 can't be made and 14 is 2 seven-cent stamps. (I guess anything that ends in a "4" will work now, too). And 15 is a multiple of 5. And 16… well 5 + 5 = 10 and 7 + 7 = 14 and 5 + 5 + 7 = 17… so I guess we can't make 16. But we can make 17.

Total Postage	How?
13	Impossible
14	7 + 7
15	10 + 5
16	Impossible
17	5 + 5 + 7

(9) Sam: How could we get 18? 7 + 7 = 14, 7 + 7 + 5 = 19, which is too big. 5 + 5 = 10, 5 + 5 + 7 = 17 and 5 + 5 + 5 + 5 = 20 so… we can't make 18.

(10) Anita: That's right! And look 18 – 5 = 13 and 18 – 7 = 11 are also impossible to make. I guess there really isn't a way you can make 18 if both 13 and 11 don't work.

[Several minutes pass, and students fill out more of the table.]

Total Postage	How?
18	Impossible
19	5 + 7 + 7
20	5 + 5 + 5 + 5
21	14 + 7
22	12 + 10
23	Impossible
24	14 + 10
25	20 + 5
26	21 + 5
27	22 + 5
28	21 + 7

(11) Sam: And we know 29 is 24 + 5. And 30 is 25 + 5. And 31 is 26 + 5. And 32 is 27 + 5. And… well… I think everything above 23 is possible if we just keep adding a five-cent stamp to some other postage. Right?

Total Postage	**How?**
23	Impossible
24	14 + 10
25	20 + 5
26	21 + 5
27	22 + 5
28	21 + 7
29	24 + 5
30	25 + 5
31	26 + 5
32	27 + 5

(12) Anita: I think you're right! We had a whole bunch of numbers in a row that worked and then if we keep adding 5 we'll get another sequence of consecutive numbers and we can add 5 to those numbers, too.

(13) Dana: Yeah, so if we have 5 consecutive numbers that work, like:

28
29
30
31
32

We can add 5 to each of those. That would get us:

28 + 5 = 33
29 + 5 = 34
30 + 5 = 35
31 + 5 = 36
32 + 5 = 37

And that's five more consecutive numbers that work! Now add 5 to each of them, and we can keep going on forever.

(14) Anita: Actually I don't even think it matters which five numbers we start with. As long as we have any five consecutive numbers that work, say:

n
$n+1$
$n+2$
$n+3$
$n+4$

Then adding 5 to each of those will get you the next 5 consecutive numbers,
$n+5$

$$(n+1)+5=n+6$$
$$(n+2)+5=n+7$$
$$(n+3)+5=n+8$$
$$(n+4)+5=n+9$$

(15) Sam: So, what's so special about 23 in this problem? It's the last one you can't get.

(16) Dana: I don't have a clue what's special about it. It doesn't seem related to 5 and 7.

(17) Sam: I wonder what would happen if we only had six-cent and seven-cent stamps…What amounts of postage could we make then?

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. What mathematics in this dialogue is most likely to be confusing for students?

3. What strategies do students in this dialogue use to find postage values that can be made? What strategies do students use to verify that postage values could not be made?

4. Students in this dialogue use a table to organize their responses. What are other ways the students could have approached this task?

5. Had the students in the dialogue started to randomly list combinations, how (if at all) would you intervene? Had the students used the table approach in the dialogue but continued making combinations up to 60 without noticing a pattern or generalizing an argument, how (if at all) would you intervene?

6. How else might students explain why every postage greater than 23 is possible?

7. How would the problem change if the two numbers were NOT relatively prime?

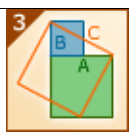
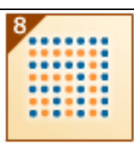
8. The students end by wondering what would happen if they changed the denominations of stamps to six-cent and seven-cent. What conjectures do you have about what will happen with six-cent and seven-cent stamps? Based on all the examples you've seen in the dialogue and Teacher Reflection Questions, what conjectures do you have about the postage amounts that are possible and impossible given a pair of denominations: m-cent and n-cent stamps?

Integer Combinations—Postage Stamps Problem (MS)

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	Some mathematical problems do not have a straightforward solution process, and diving in and exploring possibilities with a systematic approach is a way to make sense of the problem. The students make sense of the problem by organizing their data in a table (MP 5), and persevere by continuing to fill the table until a pattern emerges. It is only by exploring several cases that the students are able to get to the point where Sam sees some regularity and seizes on it—"Everything above 23 is possible if we just keep adding a five-cent stamp to some other postage" (line 11).
 Construct viable arguments and critique the reasoning of others.	Towards the end, the students form a viable argument to explain why all postage values above 23 can be made (lines 11–14). They begin by explaining why, given five consecutive numbers, you can generate all postage values greater than that (lines 12–13) and later refine the argument to show that it doesn't matter which sequence of numbers you start with as long as there are five consecutive numbers (line 14). Even earlier in the dialogue, we can find examples of constructing arguments as the students explain why certain postage values are possible while others are impossible. Showing that something is impossible is often more interesting and complicated to explain, and in the dialogue, Anita is able to come up with an argument for why 18 is impossible by subtracting 5 and 7 from 18 and seeing if those smaller postage values were possible (line 10).
 Look for and express regularity in repeated reasoning.	One example of repeated reasoning can be seen when Anita identifies that all postage greater than 12 and ending with a "2" can be made by adding 2 five-cent stamps (line 7). Sam also finds regularity by noticing that all postage values greater than 23 are possible by adding five-cent stamps (line 11). Students then apply this regularity to a sequence of consecutive numbers (line 13) and then generalize the argument using variables (line 14), realizing that their infinite pattern occurs regardless of the starting numbers as long as there are five consecutive numbers. As the students begin to conjecture at the end, they are entering a new phase of generalizing, namely, what pairs of denominations work like five-cent and seven-cent stamps do?

Integer Combinations—Postage Stamps Problem (MS)

Commentary on the Mathematics

This problem is most evidently about integer equations of the form $rM + sN = P$, where r and s are integers ≥ 0, and M and N are whole numbers. Also elicited by the problem, as seen in Anita's reasoning (line 14), are algebraic expressions and equivalence, as in $(n+1)+5 = n+6$, $(n+2)+5 = n+7$, and so on, to show that, in the cases where there are only five-cent and seven-cent stamps, once you "get" five stamps in a row, you can get all values of stamps after that point by adding five-cent stamps over and over. Even before Anita makes the argument using variables, Dana uses the numbers 28 through 32 to make an argument for why all values greater than 28 will be possible (line 13). For pre-algebra students, using numerical examples like Dana does can help them understand how all larger numbers can be made and why it is important to have five consecutive numbers that work. Alternatively, students can also generalize their explanation by looking at the simplest example of five consecutive integers, namely 1 through 5. Using a simpler sequence allows students to focus entirely on the calculation and the effect of adding +5 to each term, in the same way using n, $n+1$, $n+2$, $n+3$, and $n+4$ would. For pre-algebra students, using an archetypal example can be a precursor step to generalizing for all cases using a variable.

The problem also invites mathematical conjectures in several directions, two of which the students touch upon at the end: What pairs of stamp denominations, M and N, have the quality that, after a certain point, all stamp values can be obtained? When that "certain point" occurs for a pair M and N, is there a way to determine what that number is?

By trying different pairs of M and N, like the pair 6 and 7 that Sam suggests at the end of the dialogue, students can eventually conjecture that if M and N are relatively prime, then the "certain point" occurs, and all number values P after that point can be obtained by combinations of M and N: $rM + sN = P$, where r and s are integers ≥ 0. Some students might even find an expression to show the largest number that can't be made using combination of M and N.

<u>Evidence of the Content Standards</u>

Students are generating numbers that fit the rule $5M + 7N$. This rule is communicated in the scenario given in the original task (4.OA.C.5). To find the possible values, students are using their knowledge of multiples (lines 1–2) as well as generating composite numbers made out of 5s and 7s only (4.OA.B.4). Throughout the dialogue, the students are recording the expressions used to make the possible postage stamp amounts. Also, Anita (line 12) is able to interpret the effect of adding a five-cent stamp to a sequence of consecutive numbers (5.OA.A.2). Once a numerical argument (line 13) is made for why adding a five-cent stamp to each of five consecutive postage amounts produces the next five amounts, the same argument is made using a variable n (line 14) (6.EE.A.2).

Integer Combinations—Postage Stamps Problem (MS)

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. Find a combination of five-cent stamps and seven-cent stamps that equals 61 cents in postage. Find a combination that equals 89 cents.

2. What strategies do students in the dialogue use to find postage values that can be made?

3. Imagine that the students produced the table below showing postage values that can and can't be made using five-cent and seven-cent stamps. Is the table correct? If not, what mistakes are the students making?

Total Postage	Possible?
1	No
2	No
3	No
4	No
5	Yes
6	No
7	Yes
8	No
9	No
10	Yes
11	No
12	No
13	No
14	Yes
15	Yes
16	No
17	No
18	No
19	No
20	Yes
21	Yes

4. How do the students determine that 18 is impossible to make using only five- and seven-cent stamps? Are you convinced by the argument? Why or why not?

5. How do students determine that all postage values after 23 can be made?

Related Mathematics Tasks

1. Find two different combinations of five-cent and seven-cent stamps that can be used to make 111 cents of postage.

2. Solve the problem Sam poses at the end: Suppose the post office only sold six-cent and seven-cent stamps. What amounts of postage can be made? Explain what process you used to solve this problem.

3. Suppose the post office only sold four-cent and nine-cent stamps. What amounts of postage can be made?

4. Suppose the post office only sold six-cent and nine-cent stamps. What amount of postage can be made?

5. Suppose the post office only sold m-cent stamps and n-cent stamps. Suppose also that, above some amount of postage, all amounts of postage can be made. What can you say about m and n?

6. Suppose the post office only sold two stamp denominations that are multiples of 5. What can you say about the postage that can and can't be made? What if both stamps are multiples of n?

Integer Combinations—Postage Stamps Problem (MS)

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. What mathematics in this dialogue is most likely to be confusing for students?

   Different students may struggle with different mathematics. For students with weak number skills, they may struggle with finding which postage values are possible. Other students may struggle with understanding why, after finding five consecutive postage values that can be made, all postage values greater than that can be made. And some might have trouble with understanding why it takes a minimum of five consecutive numbers to guarantee that all postage values greater than that can also be made.

3. What strategies do students in this dialogue use to find postage values that can be made? What strategies do students use to verify that postage values could not be made?

   Students use several strategies to find postage values that work. They look for combinations of 5 and 7 that can produce the number they are looking for as in the case of 12 and 17. They also identify multiples of 5 and 7 as being possible (lines 1–2). Additionally, students realize that once a number can be made, any number greater than that and ending in the same digit would also be possible by adding 2 five-cent stamps (line 7). Eventually students find that once they have five consecutive numbers that work, they can produce the next five consecutive numbers and do so infinitely many times. The students explain this numerically (line 13) and algebraically (line 14).

   To verify that a postage, n, is not possible, students look to see if $n-5$ and $n-7$ are also impossible (line 10). If both of the smaller numbers are impossible to make, then that must be true of n as well since postage is always made from a smaller amount plus a five-cent stamp or a seven-cent stamp.

4. Students in this dialogue use a table to organize their responses. What are other ways the students could have approached this task?

   Students could have approached this task in many different ways. They might have gone about the problem in an unorganized manner, picking random numbers and seeing if they could or couldn't be made using five- and seven-cent stamps. They also could have made a table similar to the one in the dialogue but started by first filling out the multiples of 5 and 7 and then going back to see if the "holes" could be made. Student could also have made a very different sort of table, one that shows only the possible values that can be made. This table could have the number of five-cent stamps across the top and the number of seven-cent stamps going down. The cells would then show the total postage made. See example below:

7-Cent Stamps \ 5-Cent Stamps	0	1	2	3	4	5	6	7
0	0	5	10	15	20	25	30	35
1	7	12	17	22	27	32	37	42
2	14	19	24	29	34	39	44	49
3	21	26	31	36	41	46	51	56
4	28	33	38	43	48	53	58	63
5	35	40	45	50	55	60	65	70
6	42	47	52	57	62	67	72	77
7	49	54	59	64	69	74	79	84

Similarly, students with more algebra skills might write the equation: $P = 5M + 7N$ where M and N represent the number of five-cent and seven-cent stamps, respectively, and P is the value of postage. Once this equation is written, students can systematically plug various values into M and N to find what postage can be made (and what postage can't be made). Alternatively, students could use a hundreds chart and cross out numbers as they are found to be possible, leaving them with only the impossible postage values.

5. Had the students in the dialogue started to randomly list combinations, how (if at all) would you intervene? Had the students used the table approach in the dialogue but continued making combinations up to 60 without noticing a pattern or generalizing an argument, how (if at all) would you intervene?

   The response to these questions will vary greatly depending on the level of intervention you think the students need. For students who are randomly listing combinations, you might ask, "What postage can be made? What postage can't be made?" Such questions might help students realize that testing random postage values makes it difficult to answer what can and can't be made, thus pushing students towards a systematic approach.

For students who use a systematic approach and go up to high values without noticing a pattern or generalizing an argument, you might ask, "What do you notice?" If a pattern is noticed by the students, the follow-up question could be, "Why?" Alternatively, you could ask, "Do you think 1,000-cents worth of postage can be made?" Choosing a high number will force the student to think about what has been learned from previous calculations (unless the student wants to continue listing combination up to 1,000).

6. How else might students explain why every postage greater than 23 is possible?

   Students are likely to see the convincing nature of the argument that since you can get 12, you can get all numbers ending in 2, just by "adding 10 using 2 five-cent stamps" (line 7). They might then do an exhaustive argument dealing one by one with numbers ending in 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0.

7. How would the problem change if the two numbers were NOT relatively prime?

   If the two numbers weren't relatively prime (meaning they would have a greatest common factor other than 1), there would be an infinite number of postage values that can't be made. All postage values that could be made would be a multiple of that greatest common factor. For example, if stamps were sold in only six-cent and nine-cent denominations, then possible postage would include 6, 9, 12, 18, 24, etc. Values in-between these would not be possible.

8. The students end by wondering what would happen if they changed the denominations of stamps to six-cent and seven-cent. What conjectures do you have about what will happen with six-cent and seven-cent stamps? Based on all the examples you've seen in the dialogue and Teacher Reflection Questions, what conjectures do you have about the postage amounts that are possible and impossible given a pair of denominations: m-cent and n-cent stamps?

   Using six-cent and seven-cent stamps, all numbers can be made after a certain point (in this case, any value after 29). If m and n are relatively prime (i.e., m and n have a greatest common factor of 1), then all numbers can be made after a certain point (depending on m and n). Any two numbers m and n for which all numbers can be made after a certain point must be relatively prime.

Integer Combinations—Postage Stamps Problem (MS)

Possible Responses to Student Discussion Questions

1. Find a combination of five-cent stamps and seven-cent stamps that equals 61 cents in postage. Find a combination that equals 89 cents.

   61 cents of postage can be made with 8 five-cent stamps and 3 seven-cent stamps ($8 \cdot 5 + 3 \cdot 7 = 61$). 89 cents of postage can be made with 15 five-cent stamps and 2 seven-cent stamps ($15 \cdot 5 + 2 \cdot 7 = 89$).

2. What strategies do students in the dialogue use to find postage values that can be made?

   Students use several strategies to find postage values that work. They look for combinations of 5 and 7 that can produce the number they are looking for as in the case of 12 and 17. They also identify multiples of 5 and 7 as being possible (lines 1–2). Additionally, students realize that once a number can be made, any number greater than that and ending in the same digit would also be possible by adding two 5-cent stamps (line 7). Eventually, students find that once they have five consecutive numbers that work, they can produce the next five consecutive numbers and do so infinitely many times.

3. Imagine that the students produced the table below showing postage values that can and can't be made using five-cent and seven-cent stamps. Is the table correct? If not, what mistakes are the students making?

Total Postage	**Possible?**
1	No
2	No
3	No
4	No
5	Yes
6	No
7	Yes
8	No
9	No
10	Yes
11	No
12	No
13	No
14	Yes
15	Yes
16	No
17	No
18	No
19	No
20	Yes
21	Yes

The table is incorrect. Some values like 12 can be made using a combination of five-cent and seven-cent stamps. The students in this scenario only considered multiples of 5 and 7 as possible; they didn't consider combinations of 5 and 7, which are also allowed in the problem.

4. How do the students determine that 18 is impossible to make using only five- and seven-cent stamps? Are you convinced by the argument? Why or why not?

   At first, Sam tries finding a combination of postage stamps that produces 18; however, Sam is unable to find one and declares that 18 is impossible to make. Afterwards, Anita comes up with an explanation as to why 18 needs to be impossible. Anita subtracts 5 and 7 from 18 and gets 13 and 11 respectively. Since both 13 and 11 are impossible to make, that must mean 18 is impossible as well. The argument is convincing since postage is always made by adding a five-cent stamp or a seven-cent stamp to a small amount of postage. If you can't make the smaller postage, then you can't make that postage +5 or +7. Therefore, backtracking from 18 makes sense.

5. How do students determine that all postage values after 23 can be made?

   Sam notices that after a certain point, new postage is made by simply adding a five-cent stamp to a previous amount (line 11). This pattern is further explored by the students, who realize that it takes five consecutive numbers that work to ensure that all larger postage is possible (line 12–14).

Possible Responses to Related Mathematics Tasks

1. Find two different combinations of five-cent and seven-cent stamps that can be used to make 111 cents of postage.

   You can have 18 five-cent stamps and 3 seven-cent stamps ($18 \cdot 5 + 3 \cdot 7 = 111$) or 4 five-cent stamps and 13 seven-cent stamps ($4 \cdot 5 + 13 \cdot 7 = 111$).

2. Solve the problem Sam poses at the end: Suppose the post office only sold six-cent and seven-cent stamps. What amounts of postage can be made? Explain what process you used to solve this problem.

Total Postage	How?
1	Impossible
2	Impossible
3	Impossible
4	Impossible
5	Impossible
6	6
7	7
8	Impossible
9	Impossible

10	Impossible
11	Impossible
12	6 + 6
13	6 + 7
14	7 + 7
15	Impossible
16	Impossible
17	Impossible
18	6 + 6 + 6
19	6 + 6 + 7
20	6 + 7 + 7
21	7 + 7 + 7
22	Impossible
23	Impossible
24	6 + 6 + 6 + 6
25	6 + 6 + 6 + 7
26	6 + 6 + 7 + 7
27	6 + 7 + 7 + 7
28	7 + 7 + 7 + 7
29	Impossible
30	6 + 6 + 6 + 6 + 6
31	6 + 6 + 6 + 6 + 7
32	6 + 6 + 6 + 7 + 7
33	6 + 6 + 7 + 7 + 7
34	6 + 7 + 7 + 7 + 7
35	7 + 7 + 7 + 7 + 7

Every integer > 29 is possible because there are six consecutive numbers (30–35) to which you can just add a six-cent stamp in order to get the next six consecutive numbers (36–41) and so on.

3. Suppose the post office only sold four-cent and nine-cent stamps. What amounts of postage can be made?

Total Postage	**How?**
1	Impossible
2	Impossible
3	Impossible
4	4
5	Impossible
6	Impossible
7	Impossible
8	4 + 4
9	9
10	Impossible

11	Impossible
12	4 + 4 + 4
13	4 + 9
14	Impossible
15	Impossible
16	4 + 4 + 4 + 4
17	4 + 4 + 9
18	9 + 9
19	Impossible
20	4 + 4 + 4 + 4 + 4
21	4 + 4 + 4 + 9
22	4 + 9 + 9
23	Impossible
24	4 + 4 + 4 + 4 + 4 + 4
25	4 + 4 + 4 + 4 + 9
26	4 + 4 + 9 + 9
27	9 + 9 + 9

Every integer > 23 is possible because there are four consecutive numbers (24–27) to which you can just add a four-cent stamp in order to get the next four consecutive numbers (28–31) and so on.

4. Suppose the post office only sold six-cent and nine-cent stamps. What amount of postage can be made?

Total Postage	**How?**
1	Impossible
2	Impossible
3	Impossible
4	Impossible
5	Impossible
6	6
7	Impossible
8	Impossible
9	9
10	Impossible
11	Impossible
12	6 + 6
13	Impossible
14	Impossible
15	6 + 9
16	Impossible
17	Impossible
18	6 + 6 + 6 (or 9 + 9)
19	Impossible

20	Impossible
21	6 + 6 + 9
22	Impossible
23	Impossible
24	6 + 6 + 6 + 6
25	Impossible
26	Impossible
27	9 + 9 + 9

In this scenario, all integers ≥ 6 that are a multiple of 3 can be made. Unlike the previous problems, this one does not have a point at which all numbers greater than that are possible. Here, there are always gaps of two postage values that can't be between postage values that can be made (starting at 6). For example, 21 and 24 can be made while 22 and 23 can't be made.

5. Suppose the post office only sold m-cent stamps and n-cent stamps. Suppose also that, above some amount of postage, all amounts of postage can be made. What can you say about m and n?

   m and n have no common factor aside from 1. In the problems we have done so far, it was only the scenarios where the two postage stamps had no common factors other than 1 that all postage after a certain value could be made. In the problem using six-cent and nine-cent stamps, there was a common factor of 3 and there was no point past which all postage amounts were possible.

6. Suppose the post office only sold two stamp denominations that are multiples of 5. What can you say about the postage that can and can't be made? What if both stamps are multiples of n?

   If the post office only sells two stamp denominations that are multiples of 5, that means all the postage amounts that can be made will be a multiple of 5. It does *not* necessarily mean that all multiples of 5 can be made. For example, if 15-cent and 20-cent stamps are sold, you can't make 10 or 5 cents of postage. Having stamps that are multiples of 5 also means there are an infinite number of postage values you can't make. All postage values that aren't a multiple of 5 will be impossible.

   If both stamps are multiples of n, then again all the postage amounts that can be made will have to be a multiple of n. All postage that is not a multiple of n won't be possible and, therefore, an infinite number of impossible amounts exist.