Word Problem with Rational Numbers—Balancing Bars of Soap

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Word Problem with Rational Numbers—Balancing Bars of Soap* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to find the weight of a bar of soap given a balanced scale containing fractional bars of soap and weights of known quantity. Using the equivalence of the two sides of the scale and proportional reasoning, students calculate the weight of a bar of soap.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 2: Reason abstractly and quantitatively.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 5: Use appropriate tools strategically.

Target Grade Level: Grades 4–5

Target Content Domain: Numbers and Operations – Fractions

Highlighted Standard(s) for Mathematical Content

4.NF.B.4c	Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*
5.NF.B.4	Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

Math Topic Keywords: fractions, multiplication

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

On one side of a balanced scale is a full bar of soap. On the other side is $\frac{3}{4}$ of a bar of the same kind of soap and a $\frac{3}{4}$-ounce weight. Find the weight of the bar of soap.

Task Source: This task was minimally adapted from the *Progression on Expressions and Equations* document. Retrieved from http://math.arizona.edu/~ime/progressions/

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have been working with balances and understand the connection between objects being balanced and quantities being equal.

(1) Anita: If $\frac{1}{4}$ of the bar is replaced by a $\frac{3}{4}$-ounce weight, isn't the answer $4\times\left(\frac{3}{4}\right)$ of an ounce?

(2) Sam: Wait, what do you mean? Since $\frac{3}{4}$ is the same as $\frac{3}{4}$, then doesn't 1 bar weigh 1 ounce?

(3) Anita: No, that's not right… The $\frac{3}{4}$ and $\frac{3}{4}$ don't balance each other. They're on the same side of the scale, and when we add them together, they balance a full bar of soap.

(4) Sam: So if we add them together, that's $\frac{6}{4}$, which is $1\frac{1}{2}$. But $1\frac{1}{2}$ what? Bars of soap?

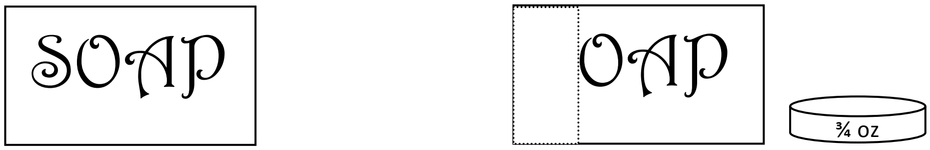
(5) Anita: No, that doesn't make sense… then a bar of soap would balance with 1½ bars of soap.

(6) Sam: Yeah, you're right, so it must be $1\frac{1}{2}$ ounces.

(7) Anita: That doesn't make sense either, does it? If a whole bar is $1\frac{1}{2}$ ounces, then half a bar would be $\frac{3}{4}$ of an ounce. So, the full bar would balance ***half*** a bar and a $\frac{3}{4}$-ounce weight.

(8) Sam: Oh yeah, you're right. *[frowns]* If I understood what you said…. Can you explain where you got your answer, Anita?

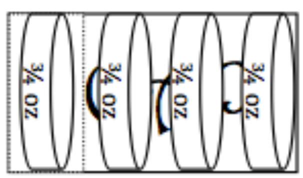
(9) Dana: Wait, I think I have something. Let me draw a picture of what I think is going on. Here's our bar of soap… *[draws]*… and the other side of the scale. Since these are balanced, they must weigh the same.



(10) Sam: That's awesome. Now what do we do? Oh wait… I see it now… You could put the $\frac{3}{4}$-ounce weight in the place of the missing part of the soap.

(11) Dana: Yup. That's even better. *[draws]*

(12) Sam: I got this now… since the missing part is $\frac{1}{4}$ of the bar and it weighs $\frac{3}{4}$ of an ounce, each of the other quarters of the bar has to weigh $\frac{3}{4}$ of an ounce. *[draws]*



There are 4 quarters total, which means $4 \times \frac{3}{4}$ of an ounce, which is 3 ounces. So, the whole bar weighs 3 ounces. Oh wait, that's exactly what you said to start Anita.

(13) Dana: That's right; but is that how much a bar of soap usually weighs?

Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. What was key in helping Sam understand why a bar of soap weighs $4 \times \frac{3}{4}$ of an ounce? What other ways might students solve this problem?

3. Sam and Anita work around two misconceptions—adding unlike things (ounces plus soap) and equating two objects that are on the same side of a scale because they have equal numbers (i.e., $\frac{3}{4}$) associated with them. What other misconceptions or obstacles might students encounter? What questions would you ask students to help them push through those obstacles or misconceptions?

4. How would the problem change if $\frac{3}{4}$ of a bar of soap balanced $\frac{1}{2}$ bar of soap and a $\frac{3}{4}$-ounce weight?

5. What if the situation had $1\frac{1}{2}$ bars of soap balancing $\frac{5}{8}$ of a bar of soap and a $\frac{1}{2}$-ounce weight?

6. Given the task in the dialogue and questions 4–5, we could employ MP 8—look for and express regularity in repeated reasoning—to write an equation that describes the general process we have followed each time. Describe that process and write an equation that represents it.

7. How is this problem the same or different from asking, "If a person travels $\frac{3}{4}$ of a mile in 15 minutes, how fast is s/he going"?

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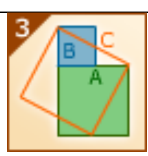
Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	Sam jumps in with a misunderstanding of the problem, and Anita has to change the thinking about how the problem is set up. Negotiating this shared understanding among multiple people is part of making sense of the problem—What do you know from reading the problem? What do you bring to the problem from prior experience? This back and forth also leads them to their ultimate entry point into the problem and this is a critical part to MP 1. All three students persevere through working with each other and ultimately solving the problem. They evaluate their approach along the way and change course when it's not working.
 Reason abstractly and quantitatively.	The students are engaging in this practice when they begin reasoning about the relationship between the quantities involved. They *decontextualize* when Anita says, "Isn't the answer $4\times\left(\frac{3}{4}\right)$?" in line 1. This symbolic representation is then manipulated to "3" and then the students *contextualize* this symbol by recognizing this 3 as ounces and as the weight of a full bar of soap. They "consider the units" and "[attend] to the meaning of the quantities, not just how to compute them" after Sam tries to add $\frac{3}{4}$ and $\frac{3}{4}$ to get $1\frac{1}{2}$.
 Construct viable arguments and critique the reasoning of others.	The back and forth in lines 1 through 8 demonstrates the interconnection of the mathematical practice standards—Anita is making sense of the problem (MP 1) while at the same time critiquing Sam's reasoning and trying to construct clear arguments to justify those critiques (MP 3).
 Use appropriate tools strategically.	In making their entry into the problem (MP 1), Dana's diagram allows Sam to make sense of the problem and ultimately leads to their solution. Sam makes strong use of this tool, especially in placing the $\frac{3}{4}$-ounce weight into the missing $\frac{1}{4}$ bar of soap.

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Commentary on the Mathematics

The task in this Illustration and the way the students solve it requires an understanding of the meaning of multiplication, in this case involving fractions. One of the $\frac{1}{4}$ bars of soap weighs $\frac{3}{4}$ of an ounce. Since there are 4 quarters in a whole bar of soap, the whole bar weighs $4\times\frac{3}{4}$. This reasoning could equally well lead to an equation such as $\frac{3}{4}=\left(\frac{1}{4}\right)x$, which the *Common Core State Standards* says students must be able to solve by the end of seventh grade. If a student were to remove a $\frac{3}{4}$ bar of soap from each side, the student would see that what remains is a $\frac{1}{4}$ bar of soap on one side and the $\frac{3}{4}$-ounce weight on the other.

Other approaches are equally valid but might stem from a different perspective. One approach could be to generate an equation of the form $x=\frac{3}{4}x+\frac{3}{4}$ where x represents the weight of the bar of soap. This equation uses the idea that "balanced" items are equal and each term represents a portion of the picture that Dana draws in line 9.

Multiplicative reasoning through third and fourth grade has involved only whole numbers, but this problem deliberately throws in fractions. Though the reasoning is no different with fractions than with whole numbers, and despite the presumption that students have a working knowledge of fractions, people's understanding of fractions tends to be more fragile than their understanding of whole numbers, and so fractions represent a distraction here. Sam tries to add $\frac{3}{4}$ of an ounce and $\frac{3}{4}$ of a bar of soap, but would almost certainly not have tried adding ounces and soap if the numbers had been whole.

Evidence of the Content Standards

The identified content standard (5.NF.B.6) is about solving real-world problems involving multiplication of fractions using a visual model or equation. While there are several ways this task could be approached, Anita (line 1) begins the dialogue with the conjecture that it is a multiplication problem and then Dana proceeds to use a visual diagram to show why this is true.

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. The students in the dialogue ultimately use a drawing to solve this problem. What other ways could you use to solve this problem?

2. Sam wanted to add $\frac{3}{4}+\frac{3}{4}$ and realized (with Anita's help) that the two cannot be added to get $\frac{6}{4}$. Why not?

3. What about Dana's last question… does 3 ounces for a bar of soap make sense?

Related Mathematics Tasks

1. On one side of a balanced scale is $\frac{3}{4}$ of a bar of soap. On the other side is $\frac{1}{2}$ of a bar of the same kind of soap and a $\frac{3}{4}$-ounce weight. How much does the bar of soap weigh?

2. On one side of a balanced scale is $1\frac{1}{2}$ bars of soap. On the other side is $\frac{5}{8}$ of a bar of the same soap and a $\frac{1}{2}$-ounce weight. How much does the bar of soap weigh?

3. Given the task in the dialogue and questions 1 and 2, describe the process you used to find the weight of one bar of soap. Then write an equation that describes that process.

4. A 2-pound block of cheese is on one side of a scale while on the other side sits $1\frac{1}{2}$ pounds of rice and $\frac{1}{2}$ of a block of the same cheese. Is the scale balanced?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. What was key in helping Sam understand why a bar of soap weighs $4 \times \frac{3}{4}$ of an ounce? What other ways might students solve this problem?

   Dana's use of a picture (line 9) was key in helping Sam understand why a quarter bar of soap weighs $\frac{3}{4}$ of an ounce. Seeing the $\frac{3}{4}$-ounce weight replace $\frac{1}{4}$ of the bar of soap helps show why multiplying the $\frac{3}{4}$-ounce weight by 4 gives the weight of the bar of soap.

   Depending on their experience, students might instead write (and solve) an algebraic equation to answer this question. $x = \left(\frac{3}{4}\right)x + \frac{3}{4}$; an alternative equation could be $\frac{1}{4}x = \frac{3}{4}$. In both equations, x represents the weight of 1 bar of soap. Please see Mathematical Overview for more details.

3. Sam and Anita work around two misconceptions—adding unlike things (ounces plus soap) and equating two objects that are on the same side of a scale because they have equal numbers (i.e., $\frac{3}{4}$) associated with them. What other misconceptions or obstacles might students encounter? What questions would you ask students to help them push through those obstacles or misconceptions?

   Discuss with colleagues to share ideas.

4. How would the problem change if $\frac{3}{4}$ of a bar of soap balanced $\frac{1}{2}$ bar of soap and a $\frac{3}{4}$-ounce weight?

   While changing the numbers changes the problem, the approach could still be the same. The $\frac{3}{4}$-ounce weight still replaces $\frac{1}{4}$ of the bar of soap, so there are still four $\frac{3}{4}$-ounce weights in a full bar, and thus, the bar weighs 3 ounces. This can also be solved with an algebraic equation, $\frac{3}{4}x = \frac{1}{2}x + \frac{3}{4}$. Recognizing that this problem is the same as the one in the dialogue except for the fact that that a $\frac{1}{4}$ bar of soap has been removed from each side of the balanced scale also explains why the bar of soap would be 3 ounces (as in the dialogue).

5. What if the situation had $1\frac{1}{2}$ bars of soap balancing $\frac{5}{8}$ of a bar of soap and a $\frac{1}{2}$-ounce weight?

   This problem will yield a different result for the weight of the bar of soap, but the approach can again be the same. The $\frac{1}{2}$-ounce weight would replace $\frac{7}{8}$ of a bar of soap. Using proportions, you can calculate that the soap would then weigh $\frac{4}{7}$ of an ounce. The approach need not change as the numbers change, so this can be a great opportunity for students to articulate a generalized statement about what they are doing each time (MP 8). See Teacher Reflection Question 6.

6. Given the task in the dialogue and questions 4–5, we could employ MP 8—look for and express regularity in repeated reasoning—to write an equation that describes the general process we have followed each time. Describe that process and write an equation that represents it.

   Each time we have found the difference in the amount of soap and replaced it with the weight. So, if there are a bars of soap on one side and b on the other with c weight, then the equation could be $(a-b)x = c$. So, $x = \frac{c}{a-b}$ where x is the weight of one bar of soap.

7. How is this problem the same or different from asking, "If a person travels $\frac{3}{4}$ of a mile in 15 minutes, how fast is s/he going"?

   On the surface, these problems seem different. However, if you convert 15 minutes to $\frac{1}{4}$ of an hour, then you might recognize that the denominators are the same and the person is traveling 3 miles per hour (mph) (since both the distance and time are scaled up by a factor of 4 resulting in 3 miles traveled in 1 hour). Both of these problems have to do with multiplicative reasoning involving fractions.

Possible Responses to Student Discussion Questions

1. The students in the dialogue ultimately use a drawing to solve this problem. What other ways could you use to solve this problem?

   One possible alternative is to write (and, of course, solve) an algebraic equation such as $x = \frac{3}{4}x + \frac{3}{4}$ where x represents the weight of the full bar of soap. Another equation that could work is that $\frac{1}{4}x = \frac{3}{4}$ where, again, x represents the weight of the full bar of soap. This second equation is more like the reasoning the students use in this Illustration.

2. Sam wanted to add $\frac{3}{4} + \frac{3}{4}$ and realized (with Anita's help) that the two cannot be added to get $\frac{6}{4}$. Why not?

   Because adding soap and ounces makes no sense at all.

3. What about Dana's last question… does 3 ounces for a bar of soap make sense?

   This question might take some investigation at home or at the store.

Possible Responses to Related Mathematics Tasks

1. On one side of a balanced scale is $\frac{3}{4}$ of a bar of soap. On the other side is $\frac{1}{2}$ of a bar of the same kind of soap and a $\frac{3}{4}$-ounce weight. How much does the bar of soap weigh?

This question is similar to the one posed in the dialogue and could be approached in a similar way—through drawings or an algebraic equation—with different numbers. The weight of the soap in this case is the same as in the dialogue: 3 ounces.

2. On one side of a balanced scale is $1\frac{1}{2}$ bars of soap. On the other side is $\frac{5}{8}$ of a bar of the same soap and a $\frac{1}{2}$-ounce weight. How much does the bar of soap weigh?

   Again, similar to the question posed in the dialogue, only these numbers result in the bar of soap weighing $\frac{4}{7}$ of an ounce.

3. Given the task in the dialogue and questions 1 and 2, describe the process you used to find the weight of one bar of soap. Then write an equation that describes that process.

   Each time we have found the difference in the amount of soap and replaced it with the weight. We then used proportions (either simple scaling up as in the dialogue, or explicitly writing out two equivalent ratios) to find the weight of the bar of soap. So, if there are a bars of soap on one side and b on the other with c weight, then the equation could be $(a-b)x = c$. So, $x = \frac{c}{a-b}$ where x is the weight of one bar of soap.

4. A 2-pound block of cheese is on one side of a scale while on the other side sits $1\frac{1}{2}$ pounds of rice and $\frac{1}{2}$ of a block of the same cheese. Is the scale balanced?

   This is very much the same type of problem, with a different unknown. Since the weight of the cheese and rice are known, this question is really about units and understanding that $\frac{1}{2}$ a pound is different from $\frac{1}{2}$ a block of cheese. This problem could be used as a launch for the task posed in the Illustration. The ultimate answer is no, because the $\frac{1}{2}$ block of cheese actually weighs 1 full pound, making the side with $\frac{1}{2}$ a block of cheese heavier than the other side of the scale.