Writing Functions—The Carnation Problem

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Writing Functions—The Carnation Problem* Illustration: This Illustration's student dialogue shows the conversation among three students writing a function for the number of pink carnations in a bouquet of t carnations given a constraint on the ratio of different colored carnations. By exploring examples of different sized bouquets, students learn the effect of the ratio constraint on the problem and eventually write a function for the number of pink flowers.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 7–8

Target Content Domain: Functions

Highlighted Standard(s) for Mathematical Content

8.F.B.4	Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Math Topic Keywords: writing functions, proportions

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Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

Asher is selling carnation flowers. He makes bouquets of different sizes by using 4 white carnations for every 3 pink carnations. How many pink carnations are there in a bouquet with t total carnations?

Task Source: This task is adapted from a released item from a Massachusetts Comprehensive Assessment System (MCAS) exam and credit is given to the Massachusetts Department of Elementary and Secondary Education.

Student Dialogue

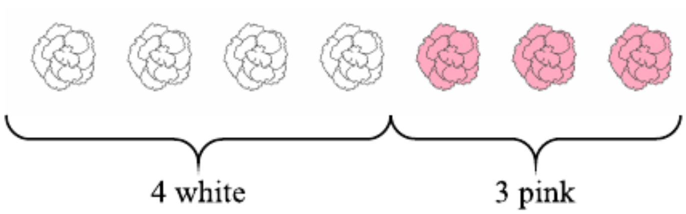
Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have been working on writing expressions for various word problems depicting real-world scenarios. They are now working on more challenging problems that involve ratio constraints.

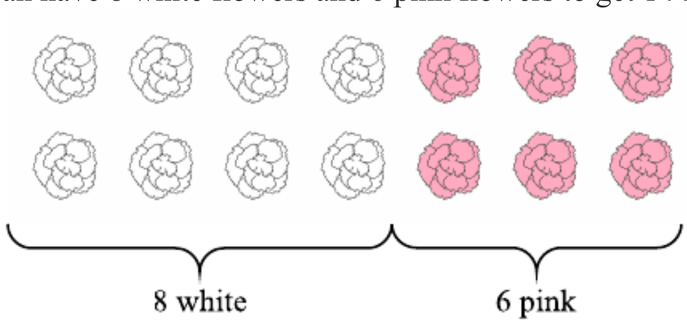
(1) Sam: This is a lot to think about all at once. How can we figure out how many pink carnations are going to be in a bouquet of t flowers?

(2) Anita: How about we start with something we do know instead. We know there are 4 white carnations for every 3 pink carnations. So let's build some bouquets with that information.

(3) Dana: Well, if we have 4 white carnations and 3 pink carnations then that's a bouquet of 7 flowers total.



Or we can have 8 white flowers and 6 pink flowers to get 14 flowers.

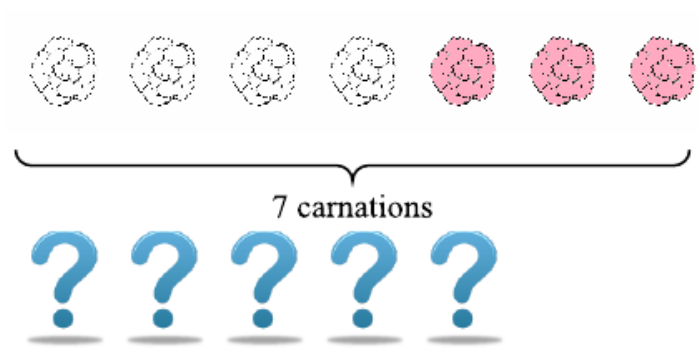


(4) Anita: It looks like any multiple of 7 will work. Look, we can even make a table with the number of white carnations, pink carnations, and total. Like this:

# of White	# of Pink	Total #
4	3	7
8	6	14
12	9	21

(5) Sam: But what if I wanted a dozen carnations. How many pink and white flowers would be in that?

(6) Dana: Well, if 4 are white and 3 are pink, that makes only 7, and we need another 5 flowers to make a dozen. What color would those have to be?



(7) Anita: You can’t split those last 5 flowers in a 4 to 3 ratio. I don’t think you can have a dozen carnations.

(8) Sam: Hmm, ok. But we still haven’t answered the question. How many pink carnations are in a bouquet of *t* flowers?

(9) Dana: Let’s look at what we have already. If there are 7 total, then there are 3 pink. If there are 14 total, then there are 6 pink.

(10) Sam: That’s fine, but how about another multiple of 7. What if we had 35 flowers? How many would be pink?

(11) Anita: There'd be 15.

(12) Sam: How about 42?

(13) Anita: 18.

(14) Sam: Wait, where are you getting these numbers?

(15) Anita: Well, if there are 35 flowers, there are 5 sets of 7, and each set has 4 white and 3 pink. So, 5 sets, each with 3 pink flowers, make 15 total pink.

(16) Dana: Let me try it with 28. There are 4 sets of 7, and there are 3 pink in every set, so there are 12 pink flowers.

(17) Anita: Ok, Sam, you try it with 707 total flowers.

(18) Sam: Well, no one is going to buy a bouquet that big! You can't even carry it! But anyway, 707 divided by 7 is 101 sets, and there are 3 pink in each set, so there are 303 pink flowers. But this still doesn't answer the question. We are supposed to find the number of pink carnations in a bouquet with *t* total flowers.

(19) Dana: Well, each time before, we divided by seven to get the number of sets and multiplied by 3 to get the number of pink carnations.

(20) Anita: So let's do that with *t* carnations. We can divide *t* by 7. That's $t \div 7$. And then we multiply by 3. That's $(t \div 7) \cdot 3$

(21) Sam: So the number of pink carnations is $(t \div 7) \cdot 3$

(22) Dana: And now we can see why the total number of all the carnations has to be a multiple of 7.

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Mathematical Practice Standards?

2. It is often useful for students to explore numerical calculations en route to writing an algebraic representation of those calculations, an algebraic expression, or an equation. For the students in the dialogue, how did their experiments with specific numbers help them get to the algebraic generalization?

3. In line 6 of the dialogue, Dana is unsure if bouquets made up of multiples other than 7 are possible. Are they possible? Why or why not?

4. If Asher were making bouquets of carnations using a ratio of 6 white carnations to 3 pink carnations, would the total number of flowers in each bouquet have to be a multiple of 9? Why or why not?

5. In lines 10–13, Anita is able to work backwards from the total number of flowers in a bouquet to the number of pink flowers. If Anita had trouble at this point, how could you help the student calculate the number of pink flowers?

6. In lines 19–21, the students are able to write an expression for the number of pink flowers given t total flowers in a bouquet. If students had trouble at this point, how could you help them calculate the number of pink flowers?

7. How many m/s in 10 km/hr? In 25 km/hr? In x km/hr?

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
1 Make sense of problems and persevere in solving them.	Students in this dialogue "analyze givens, constraints, relationships, and goals" in their exploration of the 4 white: 3 pink constraint. They begin by working with multiples of 4:3 (lines 2–4), even using a table to organize their information. Later students question what might happen if they wanted a dozen flowers (lines 5–7) which pushes students to make sense of the constraint in the problem and how it limits some bouquets from being formed.
8 Look for and express regularity in repeated reasoning.	One way mathematicians create algebraic expressions is by repeating a calculation with a few different numbers, as the students in this dialogue just did. Though the numbers keep changing, the process does not: students do the same things to each of the different numbers. Once they see the pattern, they are ready to "do the same thing" to *t* or another variable, and then they end up with an algebraic expression for the problem. In fact, they look for the "regularity in repeated reasoning"—a repeated calculation, in this case—and abstract the common elements from that repeated act, and expressed that regularity in algebraic language (MP 8). This mathematical practice is one of the central habits that we use throughout algebra. Two of its primary uses in algebra are as a mechanism for creating equations, expressions, and functions that model situations, and as a way to help us translate a word problem into an algebraic equation.

Commentary on the Mathematics

<u>"Mathematical practices" and the "practice of mathematics"</u>
There are certainly other MPs at play in this dialogue than just MP 8. This is nearly inevitable in any genuine problem-solving situation. However MP 8 is the most prominent while others can be found in the background. For example, the students discover possible bouquet arrangements (lines 3–4) and then impose a structure (line 4) on the bouquet size as a multiple of 7 (MP 7). And, although Chris is easily able to handle the mathematics of the bouquet of 707 flowers (line 18), the student is quick to note that it has to be a hypothetical case. This observation might be seen as keeping true to the sense of the problem (MP 1: sense-making) or keeping true to the context (MP 2: recontextualizing). In fact, there is no need to "distinguish" which MP or to

wonder which one to "credit" for this act. The MPs cannot provide totally disjointed baskets into which each mathematical act can be sorted. They are intended only to provide the *breadth of perspective* on what it means to engage in the practice of mathematics, by calling attention to several (not all) mathematical habits of mind.

Functions and MP 8: Look for and express regularity in repeated reasoning

Functions are defined as relations that map each element within a set (called the "domain") to *only one* element in another set (called the "range"). In algebra, functions are written as expressions that define a series of steps to be applied to each element in the domain. Since these steps are repeated regardless of the value chosen (as long as it is in the function's domain), functions lend themselves perfectly to MP 8. A function is, therefore, a "regular process" that takes an input to give you a single output.

Evidence of the Content Standards

Students in the dialogue work towards coming up with a function in which the input is the total number of carnations and the output is the number of pink carnations (8.F.B.4). The students ultimately come up with such a function in line 21.

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In this dialogue, students are not able to immediately write an expression for the number of pink carnations in a bouquet of t flowers. What strategy do they use to help them write an expression?

2. What problem arises when students try to make a bouquet with 12 flowers?

3. Can the number 5 be split in a 4:3 ratio? If yes, why can't you split 5 carnations in a 4:3 ratio?

4. Is the ratio 3:5 the same as the ratio $\frac{12}{20}$?

5. What are the steps students take to go from a total number of carnations to the number of pink carnations only? Explain why students do each step.

6. Explain why $(t \div 7) \cdot 3$ gives the number of pink carnations?

Related Mathematics Tasks

1. Using the same task as in the dialogue, how many white carnations are in a bouquet of 14 total flowers? 28 total flowers? 70 total flowers? t total flowers?

2. Using the same task as in the dialogue, does $t - \frac{t}{7} \cdot 3$ represent the number of white carnations in a bouquet of t total flowers? Explain why or why not.

3. If Asher were making bouquets of carnations using a ratio of 6 white carnations to 3 pink carnations, would the total number of flowers in each bouquet have to be a multiple of 9? Why or why not?

4. It takes Hebert 2 hours to water 5 gardens.
   A. How many gardens can he water in 4 hours? In 8 hours? In 40 hours? In *h* hours?
   B. For each question in part A, you used a repeating process. Explain what is happening in each step.

5. A. How many m/s in 10 km/hr? In 25 km/hr? In *x* km/hr?
   B. For each question in part A, you used a repeating process. Explain what is happening in each step.

6. In the 2012 Olympics, the United States won 46 gold medals and Russia won 24 gold medals. At the time, the United States had a population of approximately 314 million while Russia had a population of approximately 143 million. Based only on the information given, which country do you think was most successful at the Olympics and why?

Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Mathematical Practice Standards?

   Refer to the Mathematical Overview for notes related to this question.

2. It is often useful for students to explore numerical calculations en route to writing an algebraic representation of those calculations, an algebraic expression, or an equation. For the students in the dialogue, how did their experiments with specific numbers help them get to the algebraic generalization?

   Using numerical examples gave these students a "feel" for the process described in the problem. By trying several examples, students saw what operations were being performed on the numbers and began to notice a repeating process—different numbers but exactly the same steps in each example. Once students understand the process, they can apply those steps to a variable and write an algebraic expression.

3. In line 6 of the dialogue, Dana is unsure if bouquets made up of multiples other than 7 are possible. Are they possible? Why or why not?

   No. Only bouquets made up of multiples of 7 are possible. In this problem, bouquets consist of 4 white carnations for every 3 pink carnations. For a bouquet to have 4 pink flowers to 3 white flowers, the ratio must be $\frac{4n}{3n}$, which would make the total number of flowers $7n$. However, since bouquets are made up of only whole numbers of flowers, n must be a positive integer, which means the total number of flowers, $7n$, must be a multiple of 7.

4. If Asher were making bouquets of carnations using a ratio of 6 white carnations to 3 pink carnations, would the total number of flowers in each bouquet have to be a multiple of 9? Why or why not?

   No. In fact, bouquets could be a multiple of 3, not just 9. The reason being that the ratio $\frac{6}{3}$ can be reduced to $\frac{2}{1}$. This means that any bouquet that is a multiple of 3 with 2 white carnations for every 1 pink carnation is also possible.

5. In lines 10–13, Anita is able to work backwards from the total number of flowers in a bouquet to the number of pink flowers. If Anita had trouble at this point, how could you help the student calculate the number of pink flowers?

   Anita could be encouraged to draw a picture and find the number of pink flowers. The student shouldn't be asked to simply count but to come up with a process for finding the pink flowers using smaller sets in the bouquet. In this way, Anita will hopefully come to the method later described in line 15. Once a pattern is found (divide by 7, multiply by 3), have the student use that pattern on the tabulated results. Does taking the total number of flowers, dividing by 7, and multiplying by 3 fit the results already collected? In this way, the function relationship between the total number of flowers and the number of pink flowers can be emphasized.

6. In lines 19–21, the students are able to write an expression for the number of pink flowers given t total flowers in a bouquet. If students had trouble at this point, how could you help them calculate the number of pink flowers?

   You might ask students to write down the calculations they used to find the number of pink flowers in a bouquet of a known size. Then have students replace their numerical total of flowers with a variable. For auditory learners, students can also be asked to repeat the operations they performed on the total number of flowers out loud and then do the same operations on a variable.

7. How many m/s in 10 km/hr? In 25 km/hr? In x km/hr?

$$\frac{10\text{ km}}{\text{hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}=\frac{10\text{ km}}{60\text{ min}}$$
$$\frac{10\text{ km}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}=\frac{10\text{ km}}{3600\text{ s}}$$
$$\frac{10\text{ km}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}}=\frac{10{,}000\text{ m}}{3600\text{ s}}\approx 2.8\text{ m/s}$$

$$\frac{25\text{ km}}{\text{hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}=\frac{25\text{ km}}{60\text{ min}}$$
$$\frac{25\text{ km}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}=\frac{25\text{ km}}{3600\text{ s}}$$
$$\frac{25\text{ km}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}}=\frac{25{,}000\text{ m}}{3600\text{ s}}\approx 6.9\text{ m/s}$$
$$x\cdot\frac{1}{60}=\frac{x}{60}$$
$$\frac{x}{60}\cdot\frac{1}{60}=\frac{x}{3600}$$
$$\frac{x}{3600}\cdot\frac{1000}{1}=\frac{1000x}{3600}\approx .28x$$

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Possible Responses to Student Discussion Questions

1. In this dialogue, students are not able to immediately write an expression for the number of pink carnations in a bouquet of t flowers. What strategy do they use to help them write an expression?

   To write an expression for the number of pink carnations in a bouquet of t flowers:
   - Students begin by first making bouquets that are possible. This gives them an idea of what effect the constraints of the problem have.
   - They then pick values of t and figure out how many pink carnations will be in the bouquet. The students use this strategy because by calculating with numerical values for t they are able to figure out an exact value for pink carnations and better understand what operations are being used in the process.
   - Finally, since the same steps/operations are being performed regardless of the value chosen for t, students are able to apply those steps to t and write an expression for the number of pink carnations in the bouquet.

2. What problem arises when students try to make a bouquet with 12 flowers?

   When students try figuring out the number of pink and white carnations in a bouquet of 12, they can't make a bouquet that has 4 white carnations for every 3 pink carnations. In a bouquet of 12, they are able to make 1 set of 4 white : 3 pink carnations, equaling 7 flowers, but then 5 flowers would be left over that could not be split in a 4:3 ratio using whole numbers.

3. Can the number 5 be split in a 4:3 ratio? If yes, why can't you split 5 carnations in a 4:3 ratio?

   Yes, the number 5 can be split in a 4:3 ratio. That would be $\frac{20}{7}:\frac{15}{7}$. The reason why 5 carnations cannot be split into a 4:3 ratio is due to the implicit constraint that the number of pink and white carnations can only be whole numbers. Given this constraint, 5 cannot be split into a 4:3 ratio using only whole numbers.

4. Is the ratio 3:5 the same as the ratio $\frac{12}{20}$?

   Yes, the two ratios are the same. 3:5 can be written in fraction form as $\frac{3}{5}$. Since $\frac{3}{5}=\frac{3\cdot 4}{5\cdot 4}=\frac{12}{20}$, that means the two ratios are equivalent.

5. What are the steps students take to go from a total number of carnations to the number of pink carnations only? Explain why students do each step.

   Students take the total number of carnations and divide by 7 to get the number of sets that are 4 white to 3 pink. They then multiply the number of sets by 3 to get the total number of pink carnations since each small set has 3 pink carnations in it.

6. Explain why $(t \div 7) \cdot 3$ gives the number of pink carnations?

   t is the total number of carnations in the bouquet, which is divided by 7 to get the number of sets of flowers that are in a 4 white:3 pink ratio. The number of sets is then multiplied by 3 (there are 3 pink flowers in each small set) in order to get the total number of pink flowers.

Possible Responses to Related Mathematics Tasks

1. Using the same task as in the dialogue, how many white carnations are in a bouquet of 14 total flowers? 28 total flowers? 70 total flowers? t total flowers?

If the bouquet has 14 total flowers, $14 \div 7 = 2$ $2 \cdot 4 = 8$ There are 8 white carnations.	If the bouquet has 28 total flowers, $28 \div 7 = 4$ $4 \cdot 4 = 16$ There are 16 white carnations.	If the bouquet has 70 total flowers, $70 \div 7 = 10$ $10 \cdot 4 = 40$ There are 40 white carnations.	If the bouquet has t total flowers, $t \div 7$ $(t \div 7) \cdot 4 = \frac{t}{7} \cdot 4$ There are $\frac{t}{7} \cdot 4$ white carnations.

   An alternative way students might approach this problem is by calculating the number of pink flowers given the expression developed in the dialogue, $(t \div 7) \cdot 3$, and then subtracting out the number of pink flowers from the total number of flowers. This approach is explored in the next problem.

2. Using the same task as in the dialogue, does $t - \frac{t}{7} \cdot 3$ represent the number of white carnations in a bouquet of t total flowers? Explain why or why not.

   Yes. $t - \frac{t}{7} \cdot 3$ represents the number of white carnations. The expression $t - \frac{t}{7} \cdot 3$ is taking the total number of flowers t and subtracting out the number of pink carnations, $\frac{t}{7} \cdot 3$ or $(t \div 7) \cdot 3$. Since a bouquet is only made up of pink and white carnations, subtracting out the number of pink carnations from the total number of carnations will leave you with the number of white carnations. Alternatively, you can algebraically manipulate $t-\frac{t}{7}\cdot 3$ to show that it is equivalent to $\frac{t}{7}\cdot 4$ as shown below.

$$t-\frac{t}{7}\cdot 3=$$
$$\frac{7t}{7}-\frac{3t}{7}=$$
$$\frac{7t-3t}{7}=$$
$$\frac{4t}{7}=$$
$$\frac{t}{7}\cdot 4$$

3. If Asher were making bouquets of carnations using a ratio of 6 white carnations to 3 pink carnations, would the total number of flowers in each bouquet have to be a multiple of 9? Why or why not?

   No. In fact, bouquets can be a multiple of 3, not just 9. The reason being that the ratio $\frac{6}{3}$ can be reduced to $\frac{2}{1}$. This means that any bouquet that is a multiple of 3 with 2 white carnations for every 1 pink carnation is also possible.

4. It takes Hebert 2 hours to water 5 gardens.
   A. How many gardens can he water in 4 hours? In 8 hours? In 40 hours? In *h* hours?

In 4 hours, $\frac{4}{2}=2$ $2\cdot 5=10$ Hebert can water 10 gardens	In 8 hours, $\frac{8}{2}=4$ $4\cdot 5=20$ Hebert can water 20 gardens.	In 40 hours, $\frac{40}{2}=20$ $20\cdot 5=100$ Hebert can water 100 gardens	In *h* hours, $\frac{h}{2}$ $\frac{h}{2}\cdot 5$ Hebert can water $\frac{h}{2}\cdot 5$ gardens

   B. For each question in part A, you used a repeating process. Explain what is happening in each step.

      First, you divide the number of hours by 2 to get the number of 2-hour work periods. Then, you multiply by 5 (the number of gardens Hebert can water in a 2-hour period) to get the total number of gardens watered.

5. A. How many m/s in 10 km/hr? In 25 km/hr? In x km/hr?

$$\frac{10\text{ km}}{\text{hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}=\frac{10\text{ km}}{60\text{ min}}$$
$$\frac{10\text{ km}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}=\frac{10\text{ km}}{3600\text{ s}}$$
$$\frac{10\text{ km}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}}=\frac{10{,}000\text{ m}}{3600\text{ s}}\approx 2.8\text{ m/s}$$

$$\frac{25\text{ km}}{\text{hr}}\cdot\frac{1\text{ hr}}{60\text{ min}}=\frac{25\text{ km}}{60\text{ min}}$$
$$\frac{25\text{ km}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}=\frac{25\text{ km}}{3600\text{ s}}$$
$$\frac{25\text{ km}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}}=\frac{25{,}000\text{ m}}{3600\text{ s}}\approx 6.9\text{ m/s}$$

$$x\cdot\frac{1}{60}=\frac{x}{60}$$
$$\frac{x}{60}\cdot\frac{1}{60}=\frac{x}{3600}$$
$$\frac{x}{3600}\cdot\frac{1000}{1}=\frac{1000x}{3600}\approx .28x$$

B. For each question in part A, you used a repeating process. Explain what is happening in each step.

You first take the km/hr and divide by 60 to get km/min. Then you divide by 60 again to get km/s. Afterwards you multiply by 1000 to get m/s.

6. In the 2012 Olympics, the United States won 46 gold medals and Russia won 24 gold medals. At the time, the United States had a population of approximately 314 million while Russia had a population of approximately 143 million. Based only on the information given, which country do you think was most successful at the Olympics and why?

While the U.S. won more gold medals than Russia, if one were to take the population of a country into account, Russia had more medals won/person. The ratio of medals to persons was $\frac{46}{314}$ or .146 gold medals per million persons in the U.S., while for Russia the ratio was $\frac{24}{143}$ or .168 gold medals per million persons.