Writing Numerical Expressions—Hexagon Tables

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components mayco be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Writing Numerical Expressions—Hexagon Tables* Illustration: This Illustration's student dialogue shows the conversation among three students who are trying to figure out how many seats would be available if 57 hexagonal tables were pushed together in a single chain. Students end up discovering that more than one numerical expression can be written to calculate the number of seats, depending on how you count the seats.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 2: Reason abstractly and quantitatively.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 4: Model with mathematics.
MP 7: Look for and make use of structure.
MP 8: Look for and express regularity in repeated reasoning.

Target Grade Level: Grades 5–7

Target Content Domain: Operations and Algebraic Thinking, Expressions and Equations

Highlighted Standard(s) for Mathematical Content
5.OA.A.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
5.OA.A.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as $2 \times (8 + 7)$. Recognize that $3 \times (18932 + 921)$ is three times as large as $18932 + 921$, without having to calculate the indicated sum or product.*
7.EE.A.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."*

Math Topic Keywords: numerical expressions

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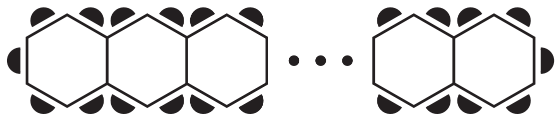
Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

A big party is being planned and everyone will sit at hexagon-shaped tables. The tables will be put together in one long line as shown below. If there are 57 tables and each side of the table fits only one person, how many guests can be seated? Write an expression to represent the number of guests that can be seated at 57 tables.



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Student Dialogue

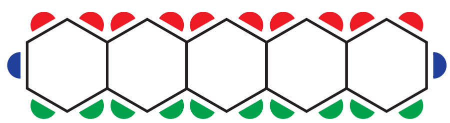
Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have been working on solving word problems with pictorial models and are learning to use their pictures to create numerical expressions. They are familiar with the order of operations for addition, subtraction, multiplication, and division and have recently learned how to work with parentheses.

(1) Dana: Oh, I know this! It's 57 times 6. Oh wait, no. When you push the tables together, the seats in between will be lost.

(2) Sam: Should we subtract 55 for the tables between? Then it would be $57 \times 6 - 55$…

(3) Anita: Let's draw a picture, but since we can't draw all 57 tables, let's just draw 5 and see what happens. *[draws 5 tables together with seats around outside]*



(4) Dana: Ok, so with 5 tables we get 22 seats: 10 along the top, 10 on the bottom, and two on the sides.

(5) Sam: No matter how many tables there are, there will always be two on the sides, but what will be the number along the top and bottom for 57 tables?

(6) Dana: Well, on the top, there are two seats for each table, so that's $2 \times 5 = 10$ on the top, and there are two of those since it's the same number on the bottom.

(7) Sam: So how would we write that as an expression?

(8) Dana: $2 \times 5 + 2 \times 5 + 2$

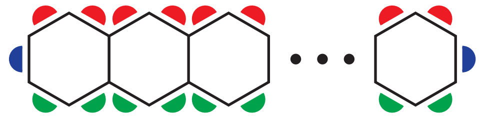
(9) Sam: Since we have 2 groups of 2×5, we could also write that as $2(2 \times 5) + 2$.

(10) Dana: So, if we had 10 tables, it would be $2 \times 10 + 2 \times 10 + 2$.

(11) Sam: Or $2(2 \times 10) + 2$ which is 44 seats.

(12) Dana: Ok, so for 57 tables, we double the number of tables, multiply by two and add 2. So the 2's stay the same and we replace the number of tables with 57: $2(2 \times 57) + 2 = 230$ seats!

(13) Sam: Yeah, that makes sense. Along the top, it's 2 × 57 seats and we multiply that by 2 for the bottom and then add 2 for the sides.



(14) Anita: I got the same answer, but I counted the seats in a different way. Using my way, I came up with the expression 4(57 – 2) + 5 × 2.

(15) Sam: How did you get that?

(16) Anita: Well, looking at the picture, I saw that the first and last tables have 5 seats and the middle tables all have 4 seats.

(17) Dana: Right, so for 57 tables, all but 2 of the tables have 4 seats.

(18) Sam: So that's 4(57 – 2) = 220 seats.

(19) Anita: Right, and each of the two tables on the end have 5 seats.

(20) Dana: Ah, so we add 5 × 2 for those and get 4(57 – 2) + 5 × 2 = 230 seats just like before!

(21) Anita: Are there other ways to count up the number of seats for the 57 tables?
[Students think and work for a little while.]

(22) Sam: Oh, I just came up with 6 × 57 – 2(57 – 1). Does that make sense?

(23) Dana: Well, 6 × 57 gives us the number of seats if the tables were not pushed together, so you must be subtracting the number of seats you lose when you push them together.

(24) Anita: Let's look back at the picture. You lose one seat for every place where the tables meet and there are 4 places where the tables meet when there are 5 tables, so there should be 56 places where they meet when there are 57 tables.

(25) Dana: Ok, so we should subtract 57 – 1 seats where the tables meet. That's 6 × 57 – (57 – 1). You had 6 × 57 – 2(57 – 1). What's wrong?

(26) Sam: Hang on! There isn't one missing seat where the tables meet. There are two missing seats—one from each table!

(27) Dana: So what does that mean?

(28) Sam: Look. For 5 tables, there would be $6 \times 5 = 30$ seats if they weren't pushed together, but we have to subtract $2 \times 4 = 8$ since we lose two seats everywhere the tables meet.

(29) Anita: So for 57 tables, there really are 56 places where they meet, but we have to multiply it by 2 for the two seats we lose in between. That's $2(57 - 1)$.

(30) Dana: So we get $6 \times 57 - 2(57 - 1)$ just like they had! Oh, and that equals 230 also!

(31) Sam: Are there any other ways to write the expression?

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

2. How else might students write an expression for the number of seats with 57 tables?

3. The students in the dialogue generalized about the case of 57 tables from their picture of 5 tables. What kinds of expressions might you see from students if they were asked to generalize about the case of t tables?

4. What if the tables were octagon-shaped? What kinds of expressions might you see from students if they were asked to generalize about the case of t tables?

5. The students in the dialogue make two early missteps before they dive into the reasoning about the problem (lines 1–2) and one later on when they forget to count 2 seats for each space between the tables (lines 25–26). What other missteps might students make in solving a problem like this, and how can you support students in working through them?

6. How else might students model the mathematics in this problem?

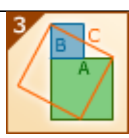
7. How would you support the students in the dialogue to explore why the expressions they created are equivalent?

8. What are some ways in which you could adapt the mathematics task to (1) elicit precise language, (2) support a variety of types of representations (visual or gestural) and how they relate to the expressions and geometric task, and (3) engage gifted students, students with special needs, and English language learners with the task?

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 Reason abstractly and quantitatively.	In the dialogue, students are making sense of quantities and their relationships in the context of solving a problem about the number of guests who can be seated at a row of hexagon-shaped tables. They create expressions to represent the quantities in the problem situation, and attend to the meaning of the quantities when they consider the meaning of different elements of the expression. They understand how the number of seats relates to the number of tables. Being able to understand what is changing and what is staying the same in these expressions is critical to writing an expression that represents any number of tables. In the dialogue, students engage in two complementary abilities that are central to MP 2—decontextualizing and contextualizing. In line 9, when Sam describes the number of seats using the expression $2(2 \times 5) + 2$ and in line 14 when Anita uses the expression $4(57 - 2) + 5 \times 2$, they "*decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents." The students in the dialogue also exhibit the "ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved," such as in lines 17 and 18, when Dana and Sam relate the expression back to the problem context, saying "Right, so for 57 tables, all but 2 of the tables have 4 seats. So that's $4(57 - 2) = 220$ seats."
 Construct viable arguments and critique the reasoning of others.	Students in the dialogue are listening critically to the explanations and arguments of others, deciding whether they make sense, and asking questions to clarify or improve the arguments. The students in the dialogue are critiquing and justifying various expressions that represent the relationship between the number of seats and the number of tables. They "make conjectures and build a logical progression of statements to explore the truth of their conjectures" in comparing different methods for approaching the problem and testing their solutions and connecting back to the context of the problem. In the dialogue, an example of students constructing arguments can be found as students explain different methods for counting the number of seats (lines 16–20).

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Model with mathematics.	In this dialogue, students write and discuss expressions to describe a situation with hexagonal tables. The students begin (line 3) by "simplify[ing] a complicated situation" from 57 tables down to 5. They are "able to identify important quantities in a practical situation and map their relationships using such tools as diagrams" when they use their diagram of 5 tables to make a conclusion about what would happen with 57 tables (lines 12–13, 17–19, 29). Throughout the dialogue, the students "routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense," as in lines 12, 20, and 30.
Look for and make use of structure.	The students in the dialogue go beyond the original task, which asked only for a number and one expression. In grade 5, the expression is pure arithmetic—no variables, no "algebra"—and so evaluating the expression is just a matter of performing the calculations and producing the number that is the other required part of the answer. When Anita (line 14) says she counted a different way, she doesn't just confirm the numerical answer; she shows her expression, *deferring* evaluation to "tell the story" of how she counted, how she *structured* her counting method. While seeing pattern is always an act of imposing structure on events, it is, by itself, too commonplace a cognitive act to "count" as MP 7. On the other hand, these students look for different ways of structuring their counting, *different* structures that they can find within the same situation, and they express these structures in their unevaluated expressions. They evaluate those expressions, too, to check that they give the same number of seats, but they recognize that the expression is the way to capture the structure that they saw. In line 12, Dana generalizes the tops + bottoms + two side seats to come up with expression for 57 tables. In line 5, Sam identified that there will always be 2 seats on the sides. Anita used a middle + end hexagons approach—observing the structure that the 2 end hexagons always contribute 5 seats each (5 × 2) and the middle hexagons always contribute 4 seats each ($4(n-2)$) (line 16). In line 22, Sam recognizes that all hexagons have 6 sides ($6n$), but you lose seats when the tables are "pushed together" ($2(n-1)$). By coming up with different methods to count the seats, students are able to see complicated things, such as the number of seats for 57 (or n) tables, as being composed of several objects.
Look for and express regularity in repeated reasoning.	The students in the dialogue model the situation from the task of 57 tables with a simpler situation with 5 tables. They observe what happens with their model of 5 tables, but, in particular, they observe their patterns of *calculation*. In lines 6–8, students realize the usefulness of representing the 10 seats along the top of the 5 tables as 2 × 5 seats. This allows the students to then generalize the process to 57 tables in line 12. Having students describe the way they counted the seats with an expression not only gives them a way to find the result but also encodes the process they used for finding the number of seats, and this kind of reasoning is critical for MP 8.

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Commentary on the Mathematics

Students in this dialogue exhibit MPs 2, 3, 4, 7, and 8 in approaching this task. The students even examine the same task in different ways, and the fact that they are able to analyze the same problem from different angles makes it clear how these mathematical practices are helping the students make sense of expressions. Furthermore, this deeper exploration into the process of making a calculation and analyzing the composition of an expression shows a way that students in grade 5 can demonstrate abstract, algebraic thinking described in MP 2 and MP 8 without the use of letters as variables. The students in the dialogue show how being able to use expressions as expressions (and not as their evaluated results) gives students a way to demonstrate these mathematical practices.

The solution method presented in this dialogue is certainly not the only way to approach this problem. Another familiar method for approaching this problem may be to make a table of values that records the total number of seats for a given number of hexagons.

Number of hexagons	1	2	3	4	5	6	7
Number of seats	6	10	14	18	22	26	30

Such a table shows that the number of seats increases at a constant rate of change, and students can use this pattern to figure out how many seats there would be with 57 hexagons. For example, from the table above, we know that 7 hexagons have 30 seats, so 50 more hexagons would add $50 \times 4 = 200$ more seats, so 57 hexagons must have 230 seats. Students who have studied such linear patterns may also express the relationship between the number of hexagonal tables (t) and the number of seats (s) with a linear equation like $s = 4t + 2$.

Students who use the table to make a pattern can exhibit the same MPs 2, 4, and 8 as the students did in the dialogue. But the way these practices are exhibited is very different. Students who make a table demonstrate the thinking of MP 4 when they use simpler cases to inform their thinking about a complex case. They exhibit MP 2 when they consider the numerical pattern in the table and how the pattern is related to the number of seats in this "real-life" situation (although it is unclear how many times in real life 57 hexagonal tables will be pushed together). And students who use a table and observe the constant rate of change and use this repeated calculation to figure out what happens in the case of 57 tables also demonstrate the use of MP 8.

The students in this dialogue do not take this approach. Instead of using the simpler cases to make a table, the students in the dialogue consider one simpler case (with 5 hexagons) and consider the *process* of finding the number of seats rather than focus on the result. The students immediately identify the total number of seats for 5 hexagons, but then their thinking goes beyond the result when they consider how their diagram led them to the result. And this kind of thinking is a critical component of how MPs 2, 4, and 8 are exhibited by these students.

This mathematical problem could also be used as a context for students to explore equivalent expressions. Students can evaluate the expressions in order to determine if they are equivalent, and they can also simplify the expressions to show how two expressions are equivalent. This problem context also provides teachers with an opportunity to support students in using a "guesscheck-generalize" strategy to build the expressions from the specific numbers they are working with as in the expressions below:

$2(2\times 5)+2$

$2(2\times 10)+2$

$2(2\times 50)+2$

$2(2\times 57)+2$

$2(2\times t)+2$

$4t+2$

Evidence of Content Standards

The students in the dialogue use parentheses in numerical expressions when they are writing expressions to describe the number of seats for a given number of hexagon tables. They also evaluate expressions using parentheses when they use the expressions to figure out how many seats are at a given number of tables (5.OA.A.1). In creating expressions that represent different methods for counting the seats, the students in the dialogue are writing expressions that record calculations with numbers, and interpret numerical expressions without evaluating them (5.OA.A.2). For example, students write the expressions $2(2 \times 57) + 2$ and $4(57 – 2) + 5 \times 2$, representing two different ways to count the number of seats at 57 hexagon tables. The students also recognize that the different forms of the expression highlight different quantities in the problem situation, and shed light on how the quantities in the problem are related (7.EE.A.2). In the expression $2(2 \times 57) + 2$, the students recognize that "$2(2 \times 57)$" corresponds to counting the top and bottom seats, and "+ 2" represents the end seats.

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Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. The students in the dialogue used a drawing of 5 tables to help them think about 57 tables. How is this helpful?

2. How does seeing Anita's expression $4(57 - 2) + 5 \times 2$ help you understand Anita's thinking?

3. How could you find the number of guests who could be seated at a row of hexagon tables given any number of tables?

4. Explain why the different expressions the students in the dialogue come up with are equivalent?

Related Mathematics Tasks

1. If there were 83 tables, how many could be seated?

2. If there were 101 tables, how many could be seated?

3. There are 154 guests seated at hexagonal tables, and no seats are empty. How many tables are there?

4. What if the tables were octagon-shaped? If there were 57 tables, how many could be seated?

5. What if the tables were arranged in different ways other than in a line? What other ways could the hexagon tables be arranged and how many could be seated?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging in the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. How else might students write an expression for the number of seats with 57 tables?

   There are many possible expressions. For example, students may notice that the first table has 6 seats and for each additional table you have to subtract one from the table before it and add 5 for the new table: 6 – (57 – 1) + 5(57 – 1).

3. The students in the dialogue generalized about the case of 57 tables from their picture of 5 tables. What kinds of expressions might you see from students if they were asked to generalize about the case of t tables?

   Students might come up with expressions similar to the numerical expressions in the dialogues such as $2(2t)+2$, $4(t-2)+5\cdot 2$, and $6t-2(t-1)$. Students may also come up with algebraic forms for other ways to count the tables such as described in Teacher Reflection Question 2: $6-(t-1)+5(t-1)$. Students may also see that all of these expressions are equivalent to the expression $4t+2$.

4. What if the tables were octagon-shaped? What kinds of expressions might you see from students if they were asked to generalize about the case of t tables?

   Students might come up with expressions similar to those found for hexagon shaped tables such as $2(3t)+2$, $6(t-2)+7\cdot 2$, $8t-2(t-1)$, $8-(t-1)+7(t-1)$, and $6t+2$.

5. The students in the dialogue make two early missteps before they dive into the reasoning about the problem (lines 1–2) and one later on when they forget to count 2 seats for each space between the tables (lines 25–26). What other missteps might students make in solving a problem like this, and how can you support students in working through them?

   Students might identify that multiplying by 6 is relevant (for the number of seats at a table), by 5 (for the number of seats at a table on the end), or by 4 (for the number of seats at a table in the middle), but might neglect to adjust by adding or subtracting to account for the actual context. Students may also be inclined to incorrectly identify the number of tables in the middle (57 – 2) or the number of places where the tables meet (57 – 1). In any case, you may find it helpful to have students test their counting method with 5 tables or another small, countable scenario such as 4 or 6 tables, and then describe how their counting method is extended to 57 tables.

6. How else might students model the mathematics in this problem?

   Students may find manipulatives such as hexagon tiles helpful for visualizing and reconfiguring the tables. They might make a chart of cases other than 5 tables and try to discover a pattern for the number of seats based on the number of tables.

7. How would you support the students in the dialogue to explore why the expressions they created are equivalent?

   You can suggest that students test the expressions with different values to see if they return equivalent results. In addition, you might ask students to explain how they know that the two expressions 2(2 × 57) + 2 and 4(57 – 2) + 5 × 2 are equivalent. Students may simplify the expressions to show that they are equivalent.

8. What are some ways in which you could adapt the mathematics task to (1) elicit precise language, (2) support a variety of types of representations (visual or gestural) and how they relate to the expressions and geometric task, and (3) engage gifted students, students with special needs, and English language learners with the task?

   (1) You might ask students to be explicit about the quantities they are representing in their expressions. You might also suggest that students describe their methods using words or a drawing before writing an expression. (2) Encourage students to use a variety of representations, and have students who use different representations share those with the class, highlighting how the expression relates to any visual representations. (3) To engage gifted students, you might consider using some of the Related Mathematics Tasks provided and, in particular, explore variations that change some of the assumptions in the task (e.g., octagon-shaped tables, different methods for arranging the tables). You might also challenge students to come up with their own variations on the mathematical task. For students with special needs and English language learners, you might consider having students represent their ideas in diagrams or pictures first before trying to translate those into expressions. Encourage students to explore the connections between their drawings and the expressions.

Possible Responses to Student Discussion Questions

1. The students in the dialogue used a drawing of 5 tables to help them think about 57 tables. How is this helpful?

   Students will likely respond that 5 tables are easier to draw than 57 tables, but press further. How does this help them to consider the context of 57 tables? Listen for students to identify that the structure is the same; there will always be 2 tables on the end with 5 seats and 2 less than the total number of tables in the middle that have 4 seats each.

2. How does seeing Anita's expression $4(57 - 2) + 5 \times 2$ help you understand Anita's thinking?

    The expression $4(57 - 2) + 5 \times 2$ allows you to consider the process by which students might have come to their answer of 230 seats. This might be helpful to you in thinking about other numbers of tables (as you could just replace the 57), and understanding their way of thinking could also help you think about other problems like parties with octagon-shaped tables.

3. How could you find the number of guests who could be seated at a row of hexagon tables given any number of tables?

    This asks students to build the logic of algebraic generalization. Students may respond with a description of the calculation they did for 5, 57, 83, 101 tables and generalize to any number of tables. For example, a student may say, "You multiply the number of tables by 4 and add," or "You multiply the number of tables by 6 and subtract 1 fewer than the number of tables 2 times."

4. Explain why the different expressions the students in the dialogue come up with are equivalent?

    Simplify the expressions $2(2 \times 57) + 2$, $4(57 - 2) + 5 \times 2$, and $6 \times 57 - 2(57 - 1)$ to show that they are equivalent. There may be opportunities to consider issues involving order of operations in working with the various expressions that represent the number of seats.

Possible Responses to Related Mathematics Tasks

1. If there were 83 tables, how many could be seated?

    334 guests. Students may use any of the expressions they created for working with 57 tables to discover the answer to this question.

2. If there were 101 tables, how many could be seated?

    406 guests.

3. There are 154 guests seated at hexagonal tables, and no seats are empty. How many tables are there?

    38 tables. This kind of "undoing" problem lets students use their expressions in a different way. A student might reason that, of the 154 guests, 2 are seated at the ends. Of the remaining 152 guests, half are seated along the "top" and half along the "bottom," so 76 guests are seated along the "top." And along the "top" of each hexagon, there is space for 2 guests, so there must be half as many tables, giving 38 tables.

4. What if the tables were octagon-shaped? If there were 57 tables, how many could be seated?

   344 people. This problem challenges students to extend their understanding to a slightly different context. Look for numerical expressions such as $2(3 \times 57) + 2$, $6(57 - 2) + 7 \times 2$, $8 \times 57 - 2(57 - 1)$, and $8 - (57 - 1) + 7(57 - 1)$.

5. What if the tables were arranged in different ways other than in a line? What other ways could the hexagon tables be arranged and how many could be seated?

   Instead of arranging the tables in a line, the tables could be "reptiled" with six hexagons forming a large hexagonal table. This is made possible by the fact that hexagons tile the plane. One other possibility would be placing a seventh hexagonal table in the middle. Either way, you get the beginning of another pattern that could grow indefinitely.